Series with de Moivre's theorem

Worked solutions

Further Maths topic practice

This topic online

1 $C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta$

$S = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta$

Show that, for $e^{i\theta} \neq 1$,

$$C + iS = \frac{e^{6i\theta} - 1}{e^{i\theta} - 1}$$

stating the first term, the common ratio and the number of terms of the series you use. (3 marks)

Pair each cosine with its sine: $\cos k\theta + i \sin k\theta = e^{ik\theta}$

$$C + iS = 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i \sin 2\theta) + \ldots + (\cos 5\theta + i \sin 5\theta)$$

$$C + iS = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + e^{4i\theta} + e^{5i\theta}$$

Geometric: $a = 1$, $r = e^{i\theta}$, $n = 6$ terms

Sum of a geometric series: $\frac{a(r^n - 1)}{(r - 1)}$

$$C + iS = \frac{(e^{6i\theta} - 1)}{(e^{i\theta} - 1)}$$

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2 a. Show that, for real k and θ,

$(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - 2k\cos\theta + k^2$

b. Hence show that

$$\frac{1}{1 - \frac{1}{2}e^{i\theta}} = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$$

(4 marks)

a) Expand the brackets

$$(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - ke^{-i\theta} - ke^{i\theta} + k^2 e^{i\theta}e^{-i\theta}$$

$$(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - k(e^{i\theta} + e^{-i\theta}) + k^2$$

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$$\boxed{(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - 2k\cos\theta + k^2}$$

b) Multiply the top and the bottom by $1 - \frac{1}{2}e^{-i\theta}$, and use $k = \frac{1}{2}$

$$\frac{1}{(1 - \frac{1}{2}e^{i\theta})} = \{(1 - \frac{1}{2}e^{-i\theta})/(1 - \cos\theta + \frac{1}{4})\}$$

$$\frac{1}{(1 - \frac{1}{2}e^{i\theta})} = \frac{(1 - \frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta)}{(\frac{5}{4} - \cos\theta)}$$

Multiply the top and the bottom by 4

$$\boxed{\frac{1}{(1 - \frac{1}{2}e^{i\theta})} = \frac{(4 - 2\cos\theta + 2i\sin\theta)}{(5 - 4\cos\theta)}}$$

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a. Show that $e^{i\theta} - 1 = 2i \sin(\frac{\theta}{2})\, e^{i\theta/2}$

b. Hence show that, for $0 < \theta < 2\pi$,

$|e^{i\theta} - 1| = 2 \sin(\frac{\theta}{2})$

(4 marks)

a) Take out a factor of $e^{i\theta/2}$

$$e^{i\theta} - 1 = e^{i\theta/2}(e^{i\theta/2} - e^{-i\theta/2})$$

$e^{ix} - e^{-ix} = 2i \sin x$ with $x = \frac{\theta}{2}$

$$\boxed{e^{i\theta} - 1 = 2i \sin(\frac{\theta}{2})\, e^{i\theta/2}}$$

b) The modulus of a product is the product of the moduli

$$|e^{i\theta} - 1| = |2i| \times |\sin(\frac{\theta}{2})| \times |e^{i\theta/2}|$$

$$|e^{i\theta} - 1| = 2|\sin(\frac{\theta}{2})| \times 1$$

$0 < \frac{\theta}{2} < \pi$, so $\sin(\frac{\theta}{2}) > 0$

$$\boxed{|e^{i\theta} - 1| = 2\sin(\frac{\theta}{2})}$$

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4 $C = 1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \ldots$

$S = -\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 3\theta + \ldots$

a. Show that $C + iS = \frac{2}{2 + e^{i\theta}}$

b. Hence show that
$C = \frac{4 + 2\cos\theta}{5 + 4\cos\theta}$
and find a similar expression for S.

(6 marks)

a) Combine the series using $\cos k\theta + i\sin k\theta = e^{ik\theta}$

$$C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} - \frac{1}{8}e^{3i\theta} + \ldots$$

Geometric: $a = 1, r = -\frac{1}{2}e^{i\theta}, |r| = \frac{1}{2} < 1$

$$C + iS = \frac{a}{(1 - r)} = \frac{1}{(1 + \frac{1}{2}e^{i\theta})}$$

$$\boxed{C + iS = \frac{2}{(2 + e^{i\theta})}}$$

b) Multiply the top and the bottom by $2 + e^{-i\theta}\}$

$$C + iS = \{2(2 + e^{-i\theta}\})/(2 + e^{i\theta})(2 + e^{-i\theta}\})\}$$

$$(2 + e^{i\theta})(2 + e^{-i\theta}\}) = 4 + 2(e^{i\theta} + e^{-i\theta}\}) + 1 = 5 + 4\cos\theta$$

$$2(2 + e^{-i\theta}\}) = 4 + 2\cos\theta - 2i\sin\theta$$

$$C + iS = \frac{(4 + 2\cos\theta - 2i\sin\theta)}{(5 + 4\cos\theta)}$$

Equate real parts and imaginary parts

$$\boxed{C = \frac{(4 + 2\cos\theta)}{(5 + 4\cos\theta)}}$$

$$\boxed{S = -\frac{2\sin\theta}{(5 + 4\cos\theta)}}$$

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5 $C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1}{4}\cos 5\theta + \frac{1}{8}\cos 7\theta + \ldots$

$S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1}{4}\sin 5\theta + \frac{1}{8}\sin 7\theta + \ldots$

a. Show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{2i\theta}}$$

b. Hence show that

$$S = \frac{6\sin\theta}{5 - 4\cos 2\theta}$$

(7 marks)

a) Combine the series

$$C + iS = e^{i\theta} + \frac{1}{2}e^{3i\theta} + \frac{1}{4}e^{5i\theta} + \ldots$$

Geometric: $a = e^{i\theta}$, $r = \frac{1}{2}e^{2i\theta}$, $|r| = \frac{1}{2} < 1$

$$C + iS = \frac{e^{i\theta}}{(1 - \frac{1}{2}e^{2i\theta})}$$

$$\boxed{C + iS = \frac{2e^{i\theta}}{(2 - e^{2i\theta})}}$$

b) Multiply the top and the bottom by $2 - e^{-2i\theta}$

$$(2 - e^{2i\theta})(2 - e^{-2i\theta}) = 4 - 2(e^{2i\theta} + e^{-2i\theta}) + 1 = 5 - 4\cos 2\theta$$

$$2e^{i\theta}(2 - e^{-2i\theta}) = 4e^{i\theta} - 2e^{-i\theta}\}$$

$$4e^{i\theta} - 2e^{-i\theta}\} = 4\cos\theta + 4i\sin\theta - 2\cos\theta + 2i\sin\theta$$

$$4e^{i\theta} - 2e^{-i\theta}\} = 2\cos\theta + 6i\sin\theta$$

$$C + iS = \frac{(2\cos\theta + 6i\sin\theta)}{(5 - 4\cos 2\theta)}$$

Equate imaginary parts

$$\boxed{S = \frac{6\sin\theta}{(5 - 4\cos 2\theta)}}$$

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6 $C = \cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$, where n is a positive integer and $\sin(\frac{\theta}{2}) \neq 0$.

a. By considering a suitable geometric series in $e^{i\theta}$, show that

$$C = \frac{\cos(\frac{(n+1)\theta}{2})\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}$$

b. Hence show that

$$\cos(\frac{2\pi}{7}) + \cos(\frac{4\pi}{7}) + \cos(\frac{6\pi}{7}) = -\frac{1}{2}$$

(9 marks)

a) Let $S = \sin\theta + \sin 2\theta + \ldots + \sin n\theta$

$$C + iS = e^{i\theta} + e^{2i\theta} + \ldots + e^{in\theta}$$

Geometric: $a = e^{i\theta}$, $r = e^{i\theta}$, n terms

$$C + iS = \frac{e^{i\theta}(e^{in\theta} - 1)}{(e^{i\theta} - 1)}$$

Use $e^{ix} - 1 = 2i\sin(\frac{x}{2})e^{ix/2}$ on the top and the bottom

$$C + iS = \frac{e^{i\theta} \times 2i\sin(n\theta/2)e^{in\theta/2}}{(2i\sin(\theta/2)e^{i\theta/2})}$$

$$C + iS = e^{i(n+1)\theta/2} \times \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

Equate real parts

$$\boxed{C = \frac{\cos(\frac{(n+1)\theta}{2})\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}}$$

b) $n = 3$, $\theta = \frac{2\pi}{7}$

$$C = \frac{\cos(\frac{4\pi}{7})\sin(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}$$

$\frac{4\pi}{7} = \pi - \frac{3\pi}{7}$, so $\cos(\frac{4\pi}{7}) = -\cos(\frac{3\pi}{7})$

$$C = -\frac{\sin(\frac{3\pi}{7})\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}$$

$2\sin A\cos A = \sin 2A$

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$$C = -\frac{\sin\left(\frac{6\pi}{7}\right)}{2\sin\left(\frac{\pi}{7}\right)}$$

$$\sin\left(\frac{6\pi}{7}\right) = \sin\left(\pi - \frac{\pi}{7}\right) = \sin\left(\frac{\pi}{7}\right)$$

$$\boxed{\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}}$$

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7 $C = 1 + 6\cos\theta + 15\cos 2\theta + 20\cos 3\theta + 15\cos 4\theta + 6\cos 5\theta + \cos 6\theta$

$S = 6\sin\theta + 15\sin 2\theta + 20\sin 3\theta + 15\sin 4\theta + 6\sin 5\theta + \sin 6\theta$

a. Show that $C + iS = (1 + e^{i\theta})^6$

b. Hence show that

$C = 64\cos^6(\frac{\theta}{2})\cos 3\theta$

and write down a similar expression for S.

c. Solve $S = 0$ for $0 < \theta < \pi$.

(8 marks)

a) Binomial coefficients of power 6: 1, 6, 15, 20, 15, 6, 1

$$C + iS = 1 + 6e^{i\theta} + 15e^{2i\theta} + 20e^{3i\theta} + 15e^{4i\theta} + 6e^{5i\theta} + e^{6i\theta}$$

$$C + iS = 1 + 6(e^{i\theta}) + 15(e^{i\theta})^2 + 20(e^{i\theta})^3 + 15(e^{i\theta})^4 + 6(e^{i\theta})^5 + (e^{i\theta})^6$$

$$\boxed{C + iS = (1 + e^{i\theta})^6}$$

b) Take out a factor of $e^{i\theta/2}$

$$1 + e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} + e^{i\theta/2}) = 2\cos(\frac{\theta}{2})\, e^{i\theta/2}$$

$$(1 + e^{i\theta})^6 = 2^6\cos^6(\frac{\theta}{2})\, e^{6i\theta/2}$$

$$(1 + e^{i\theta})^6 = 64\cos^6(\frac{\theta}{2})(\cos 3\theta + i\sin 3\theta)$$

Equate real and imaginary parts

$$\boxed{C = 64\cos^6(\frac{\theta}{2})\cos 3\theta}$$

$$\boxed{S = 64\cos^6(\frac{\theta}{2})\sin 3\theta}$$

c) $0 < \frac{\theta}{2} < \frac{\pi}{2}$, so $\cos(\frac{\theta}{2}) \neq 0$

$$\sin 3\theta = 0$$

$0 < 3\theta < 3\pi$

$$3\theta = \pi, 2\pi$$

$$\boxed{\theta = \frac{\pi}{3}, \frac{2\pi}{3}}$$

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8 $C = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \frac{\cos 4\theta}{4!} + \ldots$

$S = \sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \frac{\sin 4\theta}{4!} + \ldots$

a. Using the series expansion of e^x, which is valid for all complex x, show that $C + iS = e^{e^{i\theta}}$

b. Hence show that $C = e^{\cos\theta}\cos(\sin\theta)$ and find a similar expression for S.

c. By choosing a suitable value of θ, use your answer to part (b) to show that $1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots = \cos 1$

(7 marks)

a) Combine the series using $\cos k\theta + i\sin k\theta = e^{ik\theta}$

$$C + iS = 1 + e^{i\theta} + \frac{e^{2i\theta}}{2!} + \frac{e^{3i\theta}}{3!} + \ldots$$

Let $w = e^{i\theta}$

$$C + iS = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \ldots = e^w$$

$$\boxed{C + iS = e^{e^{i\theta}}}$$

b) Write the power as a + ib

$$e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta}$$

$$e^{e^{i\theta}} = e^{\cos\theta} \times e^{i\sin\theta}$$

$$e^{e^{i\theta}} = e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta))$$

Equate real and imaginary parts

$$\boxed{C = e^{\cos\theta}\cos(\sin\theta)}$$

$$\boxed{S = e^{\cos\theta}\sin(\sin\theta)}$$

c) Choose $\theta = \frac{\pi}{2}$: $\cos k\theta = 1, 0, -1, 0, 1, \ldots$

$$C = 1 + 0 - \frac{1}{2!} + 0 + \frac{1}{4!} + 0 - \frac{1}{6!} + \ldots$$

$$e^{\cos(\pi/2)}\cos\left(\sin\left(\frac{\pi}{2}\right)\right) = e^0\cos 1 = \cos 1$$

$$\boxed{1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots = \cos 1}$$

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9 $z = \frac{2}{3}(\cos\theta + i\sin\theta)$, where θ is real.

a. Explain why the series $1 + z + z^2 + z^3 + \ldots$ converges, and write down its sum in terms of z.

b. Use de Moivre's theorem to show that

$$\frac{2}{3}\sin\theta + \frac{4}{9}\sin 2\theta + \frac{8}{27}\sin 3\theta + \ldots = \frac{6\sin\theta}{13 - 12\cos\theta}$$

c. Show that the sum of the series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

d. Find the exact value of $\cos\theta$ for which the real part of the sum of $1 + z + z^2 + \ldots$ is equal to 1. (9 marks)

a) Geometric with ratio z

$|z| = \frac{2}{3} < 1$, so the series converges

$$\text{Sum} = \frac{1}{(1 - z)}$$

b) de Moivre: $z^k = (\frac{2}{3})^k(\cos k\theta + i\sin k\theta)$

The sine series is the imaginary part of $1 + z + z^2 + \ldots$

$$\frac{1}{(1 - z)} = \frac{1}{(1 - \frac{2}{3}e^{i\theta})} = \frac{3}{(3 - 2e^{i\theta})}$$

Multiply the top and the bottom by $3 - 2e^{-i\theta}$}

$$(3 - 2e^{i\theta})(3 - 2e^{-i\theta}\}) = 9 - 6(e^{i\theta} + e^{-i\theta}\}) + 4 = 13 - 12\cos\theta$$

$$3(3 - 2e^{-i\theta}\}) = 9 - 6\cos\theta + 6i\sin\theta$$

$$\frac{1}{(1 - z)} = \frac{(9 - 6\cos\theta + 6i\sin\theta)}{(13 - 12\cos\theta)}$$

Equate imaginary parts

$$\frac{2}{3}\sin\theta + \frac{4}{9}\sin 2\theta + \ldots = \frac{6\sin\theta}{(13 - 12\cos\theta)}$$

c) Real part of the sum

$$\text{Re} = \frac{(9 - 6\cos\theta)}{(13 - 12\cos\theta)}$$

$-1 \leq \cos\theta \leq 1$, so $9 - 6\cos\theta \geq 3$ and $13 - 12\cos\theta \geq 1$

The real part is always positive, so the sum is never purely imaginary

d) Set the real part equal to 1

$$9 - 6\cos\theta = 13 - 12\cos\theta$$

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$$6\cos\theta = 4$$

$$\boxed{\cos\theta = \frac{2}{3}}$$

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10 For a positive integer n and $\sin\theta \neq 0$,

$C = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta$

$S = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta$

a. Show that $C + iS = \dfrac{e^{i\theta}(e^{2in\theta} - 1)}{e^{2i\theta} - 1}$

b. Hence show that

$C = \dfrac{\sin 2n\theta}{2\sin\theta}$ and $S = \dfrac{\sin^2 n\theta}{\sin\theta}$

c. Hence solve the equation

$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

for $0 < \theta < \pi$.

(10 marks)

a) Combine the series

$$C + iS = e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \ldots + e^{(2n-1)i\theta}$$

Geometric: $a = e^{i\theta}$, $r = e^{2i\theta}$, n terms

$$\boxed{C + iS = \frac{e^{i\theta}(e^{2in\theta} - 1)}{(e^{2i\theta} - 1)}}$$

b) Take out half-angle factors

$$e^{2in\theta} - 1 = e^{in\theta}(e^{in\theta} - e^{-in\theta}) = 2i \sin n\theta\, e^{in\theta}$$

$$e^{2i\theta} - 1 = e^{i\theta}(e^{i\theta} - e^{-i\theta}) = 2i \sin\theta\, e^{i\theta}$$

$$C + iS = \frac{e^{i\theta} \times 2i \sin n\theta\, e^{in\theta}}{(2i \sin\theta\, e^{i\theta})}$$

$$C + iS = \frac{\sin n\theta}{\sin\theta}(\cos n\theta + i \sin n\theta)$$

Equate real parts, and use $2 \sin A \cos A = \sin 2A$

$$C = \frac{\sin n\theta \cos n\theta}{\sin\theta}$$

$$\boxed{C = \frac{\sin 2n\theta}{2\sin\theta}}$$

Equate imaginary parts

$$\boxed{S = \frac{\sin^2 n\theta}{\sin\theta}}$$

c) This is $C = 0$ with $2n - 1 = 7$, so $n = 4$

$$\frac{\sin 8\theta}{2\sin\theta} = 0$$

$$\sin 8\theta = 0$$

$0 < 8\theta < 8\pi$

$$8\theta = \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi$$

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$$\theta = \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}$$

11 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

For a positive integer n,

$C = \sum_{r=0}^{n} {}^{n}C_r \cos 2r\theta$ and $S = \sum_{r=0}^{n} {}^{n}C_r \sin 2r\theta$

a. Show that $C + iS = 2^n \cos^n \theta \, e^{in\theta}$

b. Hence write down expressions for C and S in terms of n and θ.

c. Hence find the exact values of

(i) $\sum_{r=0}^{6} {}^{6}C_r \cos(\frac{r\pi}{3})$

(ii) $\sum_{r=0}^{6} {}^{6}C_r \sin(\frac{r\pi}{3})$

(9 marks)

a) $\cos 2r\theta + i \sin 2r\theta = (e^{2i\theta})^r$

$C + iS = \sum {}^{n}C_r (e^{2i\theta})^r$

This is a binomial expansion

$C + iS = (1 + e^{2i\theta})^n$

Take out a factor of $e^{i\theta}$

$1 + e^{2i\theta} = e^{i\theta}(e^{-i\theta} + e^{i\theta}) = 2\cos\theta \, e^{i\theta}$

$$C + iS = 2^n \cos^n \theta \, e^{in\theta}$$

b) $e^{in\theta} = \cos n\theta + i \sin n\theta$; equate real and imaginary parts

$$C = 2^n \cos^n \theta \cos n\theta$$

$$S = 2^n \cos^n \theta \sin n\theta$$

c) $n = 6$ and $2\theta = \frac{\pi}{3}$, so $\theta = \frac{\pi}{6}$

$\cos^6(\frac{\pi}{6}) = (\frac{\sqrt{3}}{2})^6 = \frac{27}{64}$

(i) $C = 64 \times \frac{27}{64} \times \cos\pi = 27 \times (-1)$

$$-27$$

(ii) $S = 64 \times \frac{27}{64} \times \sin\pi = 27 \times 0$

$$0$$

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12 A small robot moves on a flat floor. It starts at the origin O and moves 1 m due east. It then turns anticlockwise through a fixed angle θ and moves $\frac{1}{2}$ m, turns anticlockwise through θ again and moves $\frac{1}{4}$ m, and so on, each move being half as long as the one before. The robot's position is modelled as a complex number, with east along the positive real axis and north along the positive imaginary axis. The diagram shows the first four moves when $\theta = \frac{\pi}{2}$.

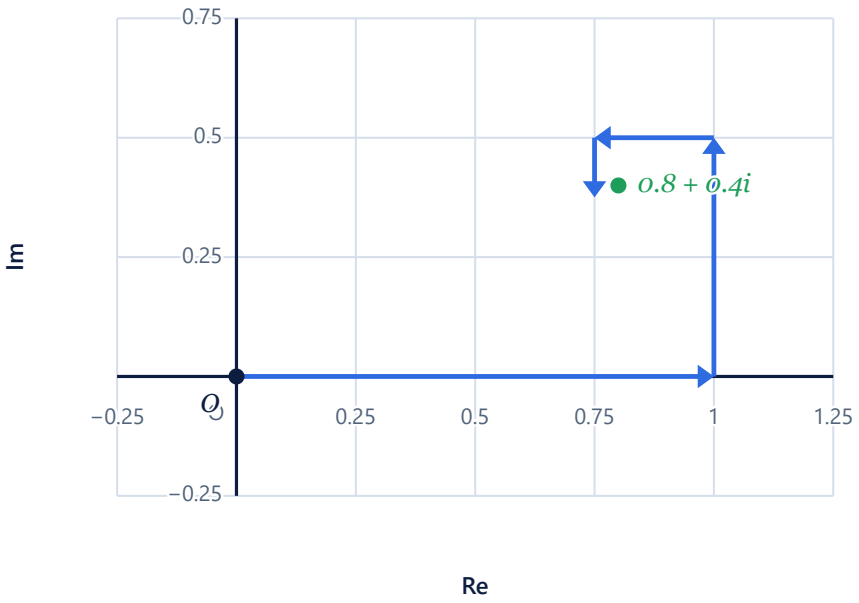
a. Explain why the point that the robot approaches is given by
$z = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots$

b. Show that
$z = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$

c. For $\theta = \frac{\pi}{2}$, find how far east and how far north of O the robot ends up.

d. Show that the distance of this point from O is $\frac{2}{\sqrt{(5 - 4\cos\theta)}}$ m, and hence find the greatest and least possible values of this distance.

e. Explain why, according to the model, the robot can never end up due north of O, whatever the value of θ.



(12 marks)

a) After k turns the robot faces at angle $k\theta$ to east

The $(k+1)$th move has length $(\frac{1}{2})^k$ and direction $k\theta$

move $= (\frac{1}{2})^k e^{ik\theta}$

Position = sum of all the moves $= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \ldots$

b) Geometric: $a = 1$, $r = \frac{1}{2}e^{i\theta}$, $|r| = \frac{1}{2} < 1$

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$$z = \frac{1}{\left(1 - \frac{1}{2}e^{i\theta}\right)} = \frac{2}{(2 - e^{i\theta})}$$

Multiply the top and the bottom by $2 - e^{-i\theta}$

$$(2 - e^{i\theta})(2 - e^{-i\theta}) = 4 - 2(e^{i\theta} + e^{-i\theta}) + 1 = 5 - 4\cos\theta$$

$$2(2 - e^{-i\theta}) = 4 - 2\cos\theta + 2i\sin\theta$$

$$\boxed{z = \frac{(4 - 2\cos\theta + 2i\sin\theta)}{(5 - 4\cos\theta)}}$$

c) $\theta = \frac{\pi}{2}$: $\cos\theta = 0$, $\sin\theta = 1$

$$z = \frac{(4 + 2i)}{5} = 0.8 + 0.4i$$

$$\boxed{\text{0.8 m east and 0.4 m north}}$$

d) $|z| = \left|\frac{2}{2 - e^{i\theta}}\right|$

$$\left|2 - e^{i\theta}\right|^2 = (2 - \cos\theta)^2 + \sin^2\theta$$

$$\left|2 - e^{i\theta}\right|^2 = 4 - 4\cos\theta + \cos^2\theta + \sin^2\theta = 5 - 4\cos\theta$$

$$\boxed{\text{distance} = \frac{2}{\sqrt{(5 - 4\cos\theta)}}\ \text{m}}$$

$-1 \leq \cos\theta \leq 1$, so $1 \leq 5 - 4\cos\theta \leq 9$

$$\text{greatest} = \frac{2}{\sqrt{1}} = 2\ \text{m}\ (\theta = 0)$$

$$\text{least} = \frac{2}{\sqrt{9}} = \frac{2}{3}\ \text{m}\ (\theta = \pi)$$

$$\boxed{\text{Greatest 2 m, least } \frac{2}{3}\text{ m}}$$

e) Due north of O needs $\mathrm{Re}(z) = 0$

$$\mathrm{Re}(z) = \frac{(4 - 2\cos\theta)}{(5 - 4\cos\theta)}$$

$4 - 2\cos\theta \geq 2$ and $5 - 4\cos\theta \geq 1$, so $\mathrm{Re}(z) > 0$ for every θ

$$\boxed{\text{The final point is always east of O's north line, never due north}}$$

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