Series with de Moivre's theorem

A-level Core Pure Mathematics | Edexcel 9FM0 | 12 questions, 88 marks

Further Maths topic practice

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Name: _____ **Date:** _____ **Score:** _____ **/ 88**

1 $C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta$

$S = \sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta$

Show that, for $e^{i\theta} \neq 1$,

$$C + iS = \frac{e^{6i\theta} - 1}{e^{i\theta} - 1}$$

stating the first term, the common ratio and the number of terms of the series you use.

(3 marks)

Answer _____

2 a. Show that, for real k and θ,

$(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - 2k\cos\theta + k^2$

b. Hence show that

$$\frac{1}{1 - \frac{1}{2}e^{i\theta}} = 4 - \frac{2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$$

(4 marks)

Answer _____

3 a. Show that $e^{i\theta} - 1 = 2i\sin\left(\frac{\theta}{2}\right)e^{i\theta/2}$

b. Hence show that, for $0 < \theta < 2\pi$,

$$|e^{i\theta} - 1| = 2\sin\left(\frac{\theta}{2}\right)$$

(4 marks)

Answer _____

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4 $C = 1 - \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta - \frac{1}{8}\cos 3\theta + \ldots$

$S = -\frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta - \frac{1}{8}\sin 3\theta + \ldots$

a. Show that $C + iS = \frac{2}{2 + e^{i\theta}}$

b. Hence show that

$C = \frac{4 + 2\cos\theta}{5 + 4\cos\theta}$

and find a similar expression for S.

(6 marks)

Answer

5 $C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1}{4}\cos 5\theta + \frac{1}{8}\cos 7\theta + \ldots$

$S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1}{4}\sin 5\theta + \frac{1}{8}\sin 7\theta + \ldots$

a. Show that

$C + iS = \frac{2e^{i\theta}}{2 - e^{2i\theta}}$

b. Hence show that

$S = \frac{6\sin\theta}{5 - 4\cos 2\theta}$

(7 marks)

Answer

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6 $C = \cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$, where n is a positive integer and $\sin(\frac{\theta}{2}) \neq 0$.

a. By considering a suitable geometric series in $e^{i\theta}$, show that

$$C = \frac{\cos(\frac{(n+1)\theta}{2})\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}$$

b. Hence show that

$$\cos(\frac{2\pi}{7}) + \cos(\frac{4\pi}{7}) + \cos(\frac{6\pi}{7}) = -\frac{1}{2}$$

(9 marks)

Answer

7 $C = 1 + 6\cos\theta + 15\cos 2\theta + 20\cos 3\theta + 15\cos 4\theta + 6\cos 5\theta + \cos 6\theta$

$S = 6\sin\theta + 15\sin 2\theta + 20\sin 3\theta + 15\sin 4\theta + 6\sin 5\theta + \sin 6\theta$

a. Show that $C + iS = (1 + e^{i\theta})^6$

b. Hence show that

$C = 64\cos^6(\frac{\theta}{2})\cos 3\theta$

and write down a similar expression for S.

c. Solve $S = 0$ for $0 < \theta < \pi$.

(8 marks)

Answer

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8 $C = 1 + \cos\theta + \frac{\cos 2\theta}{2!} + \frac{\cos 3\theta}{3!} + \frac{\cos 4\theta}{4!} + \ldots$

$S = \sin\theta + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \frac{\sin 4\theta}{4!} + \ldots$

a. Using the series expansion of e^x, which is valid for all complex x, show that

$C + iS = e^{e^{i\theta}}$

b. Hence show that $C = e^{\cos\theta}\cos(\sin\theta)$ and find a similar expression for S.

c. By choosing a suitable value of θ, use your answer to part (b) to show that

$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots = \cos 1$

(7 marks)

Answer

9 $z = \frac{2}{3}(\cos\theta + i\sin\theta)$, where θ is real.

a. Explain why the series $1 + z + z^2 + z^3 + \ldots$ converges, and write down its sum in terms of z.

b. Use de Moivre's theorem to show that
$\frac{2}{3}\sin\theta + \frac{4}{9}\sin 2\theta + \frac{8}{27}\sin 3\theta + \ldots = \frac{6\sin\theta}{13 - 12\cos\theta}$

c. Show that the sum of the series $1 + z + z^2 + z^3 + \ldots$ cannot be purely imaginary, giving a reason for your answer.

d. Find the exact value of $\cos\theta$ for which the real part of the sum of $1 + z + z^2 + \ldots$ is equal to 1.

(9 marks)

Answer

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10 For a positive integer n and $\sin\theta \neq 0$,

$C = \cos\theta + \cos 3\theta + \cos 5\theta + \ldots + \cos(2n-1)\theta$

$S = \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta$

a. Show that $C + iS = \dfrac{e^{i\theta}(e^{2in\theta} - 1)}{e^{2i\theta} - 1}$

b. Hence show that

$C = \dfrac{\sin 2n\theta}{2\sin\theta}$ and $S = \dfrac{\sin^2 n\theta}{\sin\theta}$

c. Hence solve the equation

$\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

for $0 < \theta < \pi$.

(10 marks)

Answer

11 In this question you must show all stages of your working. Solutions relying entirely on calculator technology are not acceptable.

For a positive integer n,

$C = \sum_{r=0}^{n} {}^{n}C_r \cos 2r\theta$ and $S = \sum_{r=0}^{n} {}^{n}C_r \sin 2r\theta$

a. Show that $C + iS = 2^n \cos^n \theta \, e^{in\theta}$

b. Hence write down expressions for C and S in terms of n and θ.

c. Hence find the exact values of

(i) $\sum_{r=0}^{6} {}^{6}C_r \cos\left(\frac{r\pi}{3}\right)$

(ii) $\sum_{r=0}^{6} {}^{6}C_r \sin\left(\frac{r\pi}{3}\right)$

(9 marks)

Answer

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12 A small robot moves on a flat floor. It starts at the origin O and moves 1 m due east. It then turns anticlockwise through a fixed angle θ and moves $\frac{1}{2}$ m, turns anticlockwise through θ again and moves $\frac{1}{4}$ m, and so on, each move being half as long as the one before. The robot's position is modelled as a complex number, with east along the positive real axis and north along the positive imaginary axis. The diagram shows the first four moves when $\theta = \frac{\pi}{2}$.

a. Explain why the point that the robot approaches is given by

$$z = 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots$$

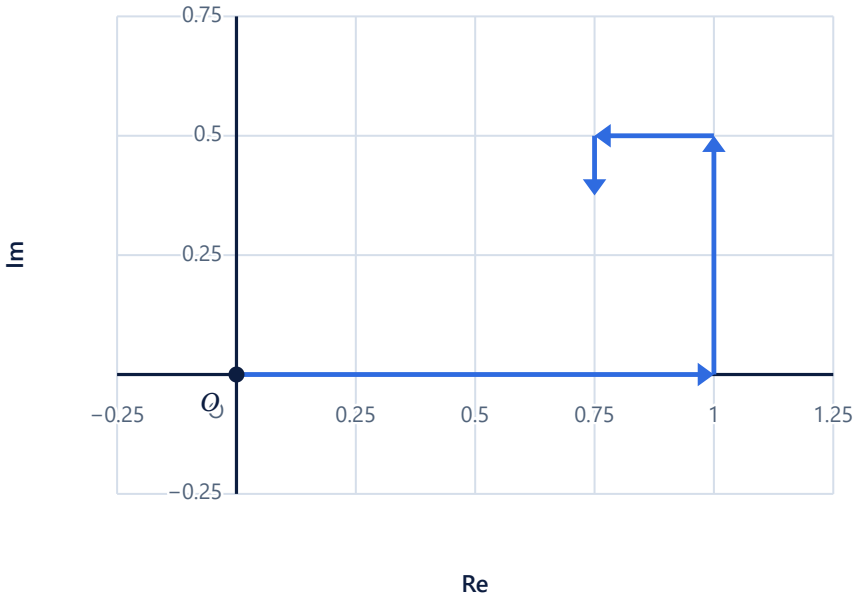
b. Show that

$$z = \frac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$$

c. For $\theta = \frac{\pi}{2}$, find how far east and how far north of O the robot ends up.

d. Show that the distance of this point from O is $\frac{2}{\sqrt{(5 - 4\cos\theta)}}$ m, and hence find the greatest and least possible values of this distance.

e. Explain why, according to the model, the robot can never end up due north of O, whatever the value of θ.



(12 marks)

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Answer

End of questions | Total 88 marks

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