Simple harmonic motion and damped oscillations

Worked solutions

Further Maths topic practice

This topic online

1 A particle P moves along the x-axis. At time t seconds its displacement from the origin O is x metres, where

$$\frac{d^2x}{dt^2} = -16x$$

When $t = 0$, P is at rest at the point where $x = 0.3$

a. Write down the general solution of the differential equation.

b. Find x in terms of t.

c. State the period of the motion and find the maximum speed of P.

(5 marks)

a)

Auxiliary equation $m^2 + 16 = 0$

$m = \pm 4i$

$$x = A\cos 4t + B\sin 4t$$

b)

$t = 0, x = 0.3$

$A = 0.3$

$$\frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t$$

$t = 0, \frac{dx}{dt} = 0$

$4B = 0$

$B = 0$

$$x = 0.3\cos 4t$$

c)

$$\text{Period} = \frac{2\pi}{4}$$

$$\text{Period} = \frac{\pi}{2}\text{ s}$$

$$\frac{dx}{dt} = -1.2\sin 4t$$

Greatest when $\sin 4t = \pm 1$

$$\text{Maximum speed} = 1.2\text{ m s}^{-1}$$

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2 A particle P moves on a straight line. At time t seconds, the displacement of P from a fixed point O on the line is x metres. The acceleration of P is always directed towards O and has magnitude $25|x|$ m s^{-2}.

When $t = 0$, P passes through O moving in the direction of increasing x with speed 3 m s^{-1}.

a. Write down a differential equation for the motion and hence show that $x = 0.6 \sin 5t$

b. State the amplitude of the motion.

c. Find the exact time at which P first reaches the point where $x = 0.3$, and the exact speed of P at that moment.

(6 marks)

a)

Acceleration is towards O, so it is negative when $x > 0$

$$\frac{d^2x}{dt^2} = -25x$$

Auxiliary equation $m^2 + 25 = 0$

$$m = \pm 5i$$

$$x = A\cos 5t + B\sin 5t$$

$t = 0, x = 0$

$$A = 0$$

$$\frac{dx}{dt} = 5B\cos 5t$$

$t = 0, \frac{dx}{dt} = 3$

$$5B = 3$$

$$B = 0.6$$

$$\boxed{x = 0.6\sin 5t}$$

b)

$$\boxed{\text{Amplitude} = 0.6\text{ m}}$$

c)

$$0.6\sin 5t = 0.3$$

$$\sin 5t = \frac{1}{2}$$

Smallest positive solution

$$5t = \frac{\pi}{6}$$

$$\boxed{t = \frac{\pi}{30}\text{ s}}$$

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$$\frac{dx}{dt} = 3\cos 5t$$

$$\frac{dx}{dt} = 3\cos\frac{\pi}{6}$$

$$\text{Speed} = \frac{3\sqrt{3}}{2}\ \text{m s}^{-1}$$

3 Each of the following differential equations models the displacement x of an oscillating system at time t.

For each equation, state whether the system is lightly damped, critically damped or heavily damped, and write down the general solution.

a. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

b. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 5x = 0$

c. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 0$

(6 marks)

a)

Auxiliary equation $m^2 + 6m + 9 = 0$

$$(m+3)^2 = 0$$

Repeated root $m = -3$, so critical damping

$$x = (A + Bt)e^{-3t}$$

b)

Auxiliary equation $m^2 + 6m + 5 = 0$

$$(m+1)(m+5) = 0$$

Two distinct real roots $m = -1$ and $m = -5$, so heavy damping

$$x = Ae^{-t} + Be^{-5t}$$

c)

Auxiliary equation $m^2 + 6m + 13 = 0$

$$m = \frac{(-6 \pm \sqrt{(36 - 52)})}{2}$$

$$m = -3 \pm 2i$$

Complex roots with negative real part, so light damping

$$x = e^{-3t}(A\cos 2t + B\sin 2t)$$

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4 The displacement x metres of a particle P from a fixed point O at time t seconds is modelled by

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + 25x = 0$$

where k is a positive constant.

a. Find the range of values of k for which the motion is lightly damped, and the value of k for which it is critically damped.

Given that k takes the value for critical damping, and that when $t = 0$, $x = 0.2$ and P is at rest,

b. find x in terms of t,

c. show that, according to the model, P never passes through O.

(7 marks)

a)

Auxiliary equation $m^2 + 2km + 25 = 0$

$\text{Discriminant} = (2k)^2 - 4 \times 25$

$\text{Discriminant} = 4k^2 - 100$

Light damping: complex roots, so discriminant < 0

$4k^2 < 100$

$k^2 < 25$

$\boxed{0 < k < 5}$

Critical damping: discriminant $= 0$

$\boxed{k = 5}$

b)

$m^2 + 10m + 25 = 0$

$(m + 5)^2 = 0$

$x = (A + Bt)e^{-5t}$

$t = 0, x = 0.2$

$A = 0.2$

$\frac{dx}{dt} = Be^{-5t} - 5(A + Bt)e^{-5t}$

$t = 0, \frac{dx}{dt} = 0$

$B - 5A = 0$

$B = 1$

$\boxed{x = (0.2 + t)e^{-5t}}$

c)

For $t \geq 0$, $0.2 + t > 0$ and $e^{-5t} > 0$

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So $x > 0$ for all $t \geqslant 0$

x is never 0, so P never passes through O

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5 The pointer on a set of kitchen scales is flicked so that it moves away from its rest position. At time t seconds after it is flicked, the displacement of the pointer from its rest position is x mm, where

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$$

When $t = 0$, $x = 0$ and the pointer is moving with speed 60 mm s^{-1} in the direction of increasing x.

a. Explain why the motion of the pointer is heavily damped.

b. Find x in terms of t.

c. Find, according to the model, the greatest displacement of the pointer from its rest position.

d. Show that, according to the model, the pointer does not return to its rest position at any time $t >$ 0

(8 marks)

a)

Auxiliary equation $m^2 + 5m + 4 = 0$

$(m + 1)(m + 4) = 0$

$m = -1$ or $m = -4$

Two distinct real roots, so the motion is heavily damped

b)

$x = Ae^{-t} + Be^{-4t}$

$t = 0, x = 0$

$A + B = 0$

$\frac{dx}{dt} = -Ae^{-t} - 4Be^{-4t}$

$t = 0, \frac{dx}{dt} = 60$

$-A - 4B = 60$

Add the two equations

$-3B = 60$

$B = -20$

$A = 20$

$x = 20e^{-t} - 20e^{-4t}$

c)

Greatest displacement when $\frac{dx}{dt} = 0$

$-20e^{-t} + 80e^{-4t} = 0$

$e^{3t} = 4$

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$$t = \frac{1}{3}\ln 4$$

$$x = 20(4^{-1/3} - 4^{-4/3})$$

Greatest displacement = 9.45 mm

d)

$x = 20e^{-4t}(e^{3t} - 1)$

For $t > 0$, $e^{3t} > 1$ and $e^{-4t} > 0$

$x > 0$ for all $t > 0$, so the pointer never returns to its rest position

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6 The front suspension of a mountain bike is being tested. At time t seconds, the displacement of the front axle above its equilibrium position is h cm, where

$$\frac{d^2h}{dt^2} + 2\frac{dh}{dt} + 10h = 0$$

When $t = 0$, the axle is 12 cm below its equilibrium position and is moving upwards with speed 6 cm s^{-1}.

a. Find h in terms of t.

b. Find the displacement of the axle 0.5 seconds after the test starts.

c. Find the time at which the axle first passes through its equilibrium position.

d. Describe the motion of the axle as t becomes large.

(9 marks)

a)

Auxiliary equation $m^2 + 2m + 10 = 0$

$$m = \frac{(-2 \pm \sqrt{(4 - 40)})}{2}$$

$$m = -1 \pm 3i$$

$$h = e^{-t}(A \cos 3t + B \sin 3t)$$

$t = 0, h = -12$

$$A = -12$$

$$\frac{dh}{dt} = e^{-t}((3B - A)\cos 3t - (3A + B)\sin 3t)$$

$t = 0, \frac{dh}{dt} = 6$

$$3B - A = 6$$

$$3B + 12 = 6$$

$$B = -2$$

$$\boxed{h = -e^{-t}(12 \cos 3t + 2 \sin 3t)}$$

b)

$$h = -e^{-0.5}(12 \cos 1.5 + 2 \sin 1.5)$$

$$h = -1.72...$$

$$\boxed{\text{1.72 cm below the equilibrium position}}$$

c)

$$12 \cos 3t + 2 \sin 3t = 0$$

$$\tan 3t = -6$$

First positive solution has $3t$ in the second quadrant

$$3t = \pi - \arctan 6$$

$t = 0.579$ s

d)

The axle keeps oscillating (period $\frac{2\pi}{3}$ s) but the amplitude decays like e^{-t}, so it settles at its equilibrium position

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7 A component in a machine is driven by a motor. At time t seconds its displacement from a fixed point is x cm, where

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\cos t$$

a. Find the complementary function for this differential equation.

b. Find a particular integral.

c. Hence write down the general solution.

d. Describe the motion of the component for large values of t, giving the exact amplitude of the motion.

(9 marks)

a)

$m^2 + 2m + 5 = 0$

$m = -1 \pm 2i$

$\boxed{x = e^{-t}(A\cos 2t + B\sin 2t)}$

b)

Try $x = \lambda\cos t + \mu\sin t$

$\frac{dx}{dt} = -\lambda\sin t + \mu\cos t$

$\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$

Substitute

$(-\lambda + 2\mu + 5\lambda)\cos t + (-\mu - 2\lambda + 5\mu)\sin t = 10\cos t$

Compare coefficients

$4\lambda + 2\mu = 10$

$4\mu - 2\lambda = 0$

$\lambda = 2\mu$

$10\mu = 10$

$\mu = 1, \lambda = 2$

$\boxed{x = 2\cos t + \sin t}$

c)

$\boxed{x = e^{-t}(A\cos 2t + B\sin 2t) + 2\cos t + \sin t}$

d)

As $t \to \infty$, $e^{-t} \to 0$, so the CF dies away

$x \approx 2\cos t + \sin t$

Amplitude $= \sqrt{(2^2 + 1^2)}$

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Oscillations of period 2π s with amplitude $\sqrt{5}$ cm, whatever the initial conditions

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8 A small weight hangs on a spring. At time t seconds, the displacement of the weight above its equilibrium position is x cm. The motion is modelled by

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + kx = 0$$

where k is a positive constant.

The weight is pulled down 6 cm below its equilibrium position and released from rest at time $t = 0$. Immediately after it is released, the acceleration of the weight is 174 cm s^{-2} upwards.

a. Find the value of k.

b. Find x in terms of t.

c. Find the displacement and the velocity of the weight when $t = 0.4$

d. Explain what the model predicts for the motion of the weight in the long term.

(10 marks)

a)

At $t = 0$: $x = -6$, $\frac{dx}{dt} = 0$, $\frac{d^2x}{dt^2} = 174$

$$174 + 4(0) + k(-6) = 0$$

$$\boxed{k = 29}$$

b)

$$m^2 + 4m + 29 = 0$$

$$m = \frac{(-4 \pm \sqrt{(16 - 116)})}{2}$$

$$m = -2 \pm 5i$$

$$x = e^{-2t}(A\cos 5t + B\sin 5t)$$

$t = 0$, $x = -6$

$$A = -6$$

$$\frac{dx}{dt} = e^{-2t}((5B - 2A)\cos 5t - (5A + 2B)\sin 5t)$$

$t = 0$, $\frac{dx}{dt} = 0$

$$5B - 2A = 0$$

$$B = \frac{2A}{5}$$

$$B = -2.4$$

$$\boxed{x = -e^{-2t}(6\cos 5t + 2.4\sin 5t)}$$

c)

$$x = -e^{-0.8}(6\cos 2 + 2.4\sin 2)$$

$$\boxed{x = 0.141 \text{ cm}}$$

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Use $A = -6$, $B = -2.4$ in $\frac{dx}{dt}$

$$5A + 2B = -34.8$$

$$\frac{dx}{dt} = 34.8e^{-2t}\sin 5t$$

$$\frac{dx}{dt} = 34.8e^{-0.8}\sin 2$$

Velocity = 14.2 cm s^{-1} upwards

d)

$e^{-2t} \to 0$ so $x \to 0$: the oscillations die away and the weight comes to rest at its equilibrium position

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9 During a charity walk, people crossing a footbridge fall into step, which makes the bridge sway. At time t seconds, the sideways displacement of the centre of the bridge from its equilibrium position is x cm, where

$$\frac{d^2x}{dt^2} + 4x = 3 \sin 2t$$

a. Find the value of the constant λ for which $x = \lambda t \cos 2t$ is a particular integral of the differential equation.

b. Hence find the general solution of the differential equation.

When $t = 0$, the centre of the bridge is at rest in its equilibrium position.

c. Find the particular solution.

d. Find, according to the model, the displacement of the centre of the bridge when $t = 10$

e. Comment on the suitability of the model for large values of t.

(11 marks)

a)

$$x = \lambda t \cos 2t$$

$$\frac{dx}{dt} = \lambda \cos 2t - 2\lambda t \sin 2t$$

$$\frac{d^2x}{dt^2} = -2\lambda \sin 2t - 2\lambda \sin 2t - 4\lambda t \cos 2t$$

$$\frac{d^2x}{dt^2} = -4\lambda \sin 2t - 4\lambda t \cos 2t$$

Substitute into the equation

$$-4\lambda \sin 2t - 4\lambda t \cos 2t + 4\lambda t \cos 2t = 3 \sin 2t$$

$$-4\lambda = 3$$

$$\boxed{\lambda = -\frac{3}{4}}$$

b)

Auxiliary equation $m^2 + 4 = 0$

$$m = \pm 2i$$

$$\boxed{x = A \cos 2t + B \sin 2t - \frac{3}{4} t \cos 2t}$$

c)

$t = 0, x = 0$

$$A = 0$$

$$\frac{dx}{dt} = 2B \cos 2t - \frac{3}{4} \cos 2t + \frac{3}{2} t \sin 2t$$

$t = 0, \frac{dx}{dt} = 0$

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$$2B - \frac{3}{4} = 0$$

$$B = \frac{3}{8}$$

$$\boxed{x = \frac{3}{8}\sin 2t - \frac{3}{4}t\cos 2t}$$

d)

$$x = \frac{3}{8}\sin 20 - \frac{3}{4}(10)\cos 20$$

$$\boxed{x = -2.72 \text{ cm}}$$

e)

The amplitude grows like t without limit, which is unrealistic: damping would limit the sway (or the bridge would fail), so the model is not suitable for large t

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10 A particle P of mass 2 kg moves on a smooth horizontal surface, attached to a spring and a damper. At time t seconds, the displacement of P from its equilibrium position O is x metres. The spring exerts a force of magnitude $50|x|$ N on P, directed towards O. The damper exerts a resistance of magnitude λv N, where v m s^{-1} is the speed of P and λ is a positive constant.

a. Show that $2\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + 50x = 0$

b. Find the value of λ for which the motion is critically damped.

Given that λ takes this value, and that when $t = 0$, $x = 0.5$ and P is moving towards O with speed 4 m s^{-1},

c. find x in terms of t,

d. show that P passes through O, and find the greatest distance of P from O on the other side of O.

The damper is replaced so that $\lambda = 30$

e. Describe how this changes the type of motion.

(12 marks)

a)

Newton's second law, x increasing away from O

$$2\frac{d^2x}{dt^2} = -50x - \lambda\frac{dx}{dt}$$

$$\boxed{2\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + 50x = 0}$$

b)

Auxiliary equation $2m^2 + \lambda m + 50 = 0$

Critical damping: repeated root, so discriminant = 0

$$\lambda^2 - 4 \times 2 \times 50 = 0$$

$$\lambda^2 = 400$$

$$\boxed{\lambda = 20}$$

c)

$$2m^2 + 20m + 50 = 0$$

$$(m + 5)^2 = 0$$

$$x = (A + Bt)e^{-5t}$$

$t = 0$, $x = 0.5$

$$A = 0.5$$

$$\frac{dx}{dt} = (B - 5A - 5Bt)e^{-5t}$$

$t = 0$, moving towards O, so $\frac{dx}{dt} = -4$

$$B - 2.5 = -4$$

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$$B = -1.5$$

$$\boxed{x = (0.5 - 1.5t)e^{-5t}}$$

d)

$$0.5 - 1.5t = 0$$

$$t = \frac{1}{3}$$

$x = 0$ at $t = \frac{1}{3}$, then $x < 0$ for $t > \frac{1}{3}$

$$\frac{dx}{dt} = (-1.5 - 2.5 + 7.5t)e^{-5t}$$

$$\frac{dx}{dt} = (7.5t - 4)e^{-5t}$$

$$\frac{dx}{dt} = 0$$

$$t = \frac{8}{15}$$

$$x = (0.5 - 0.8)e^{-8/3}$$

$$x = -0.3e^{-8/3}$$

$$\boxed{\text{Greatest distance} = 0.0208 \text{ m}}$$

e)

$$m^2 + 15m + 25 = 0$$

$$\text{Discriminant} = 225 - 100 = 125$$

$$\boxed{\begin{array}{l}\text{Two distinct negative real roots: the motion is heavily} \\ \text{damped, with no oscillation}\end{array}}$$

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11 A marker buoy on a lake bobs up and down after a boat passes. At time t seconds, the height of the buoy above its equilibrium level is x metres. The motion is modelled by

$$\frac{d^2x}{dt^2} + 0.4\frac{dx}{dt} + 4.04x = 0$$

When $t = 0$, $x = 0.5$ and $\frac{dx}{dt} = -0.1$

a. Show that $x = 0.5e^{-0.2t}\cos 2t$

b. Show that successive highest points of the buoy are π seconds apart, and find the ratio of the height of one highest point above the equilibrium level to the height of the one before it.

c. Using $|x| \leq 0.5e^{-0.2t}$, find a value of T such that, according to the model, the buoy stays within 1 cm of its equilibrium level for all $t > T$.

d. Give one criticism of the model.

(11 marks)

a)

$$m^2 + 0.4m + 4.04 = 0$$

$$m = \frac{(-0.4 \pm \sqrt{(0.16 - 16.16)})}{2}$$

$$m = -0.2 \pm 2i$$

$$x = e^{-0.2t}(A\cos 2t + B\sin 2t)$$

$t = 0$, $x = 0.5$

$$A = 0.5$$

$$\frac{dx}{dt} = e^{-0.2t}((2B - 0.2A)\cos 2t - (2A + 0.2B)\sin 2t)$$

$t = 0$, $\frac{dx}{dt} = -0.1$

$$2B - 0.1 = -0.1$$

$$B = 0$$

$$\boxed{x = 0.5e^{-0.2t}\cos 2t}$$

b)

$$\frac{dx}{dt} = -0.1e^{-0.2t}(\cos 2t + 10\sin 2t)$$

Stationary points where $\tan 2t = -0.1$

Solutions for $2t$ are π apart, so solutions for t are $\frac{\pi}{2}$ apart

Maxima and minima alternate, so successive maxima are π s apart

$$x(t + \pi) = 0.5e^{-0.2(t + \pi)}\cos(2t + 2\pi)$$

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$x(t + T) = e^{-0.2T} x(t)$

Ratio $= e^{-0.2T} = 0.533$

c)

$|\cos 2t| \leq 1$, so $|x| \leq 0.5e^{-0.2t}$

$0.5e^{-0.2t} < 0.01$

$e^{0.2t} > 50$

$t > 5 \ln 50$

$t > 19.6$ s

d)

The model ignores further disturbances (wind, waves from other boats) that would keep the buoy moving

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12 A parcel is placed gently on the platform of a spring weighing scale. At time t seconds after the parcel is placed on it, the platform has moved a distance x cm down from its unloaded position, where

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 20x = 40$$

When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$

a. Find the general solution of the differential equation.

b. Find the particular solution.

c. Show that $\frac{dx}{dt} = 10e^{-2t}\sin 4t$

d. Find the exact value of the greatest distance that the platform moves down, and state the position at which the platform eventually settles.

The manufacturer changes the damping so that the model becomes $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 20x = 40$, where c is a positive constant.

e. Find the exact value of c for which the motion is critically damped, and explain why the manufacturer might prefer this.

(12 marks)

a)

$m^2 + 4m + 20 = 0$

$m = -2 \pm 4i$

CF: $x = e^{-2t}(A\cos 4t + B\sin 4t)$

PI: try $x = k$

$20k = 40$

$k = 2$

$$x = e^{-2t}(A\cos 4t + B\sin 4t) + 2$$

b)

$t = 0, x = 0$

$A + 2 = 0$

$A = -2$

$\frac{dx}{dt} = e^{-2t}((4B - 2A)\cos 4t - (4A + 2B)\sin 4t)$

$t = 0, \frac{dx}{dt} = 0$

$4B - 2A = 0$

$4B + 4 = 0$

$B = -1$

$$x = 2 - e^{-2t}(2\cos 4t + \sin 4t)$$

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c)

$4B - 2A = 0$

$4A + 2B = -8 - 2 = -10$

$$\frac{dx}{dt} = 10e^{-2t}\sin 4t$$

d)

$\frac{dx}{dt} > 0$ for $0 < 4t < \pi$ and $\frac{dx}{dt} < 0$ just after, so maximum at

$4t = \pi$

$t = \frac{\pi}{4}$

$x = 2 - e^{-\pi/2}(2\cos\pi + \sin\pi)$

Greatest distance $= 2 + 2e^{-\pi/2}$ cm

Later maxima are smaller because of the factor e^{-2t}

As $t \to \infty$, $e^{-2t} \to 0$

The platform settles 2 cm below its unloaded position

e)

Critical damping: $m^2 + cm + 20 = 0$ has a repeated root

$c^2 - 4 \times 20 = 0$

$c^2 = 80$

$c = 4\sqrt{5}$

No overshoot or oscillation, so the platform settles at its reading without bouncing

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