Volumes of revolution (Year 2)

Worked solutions

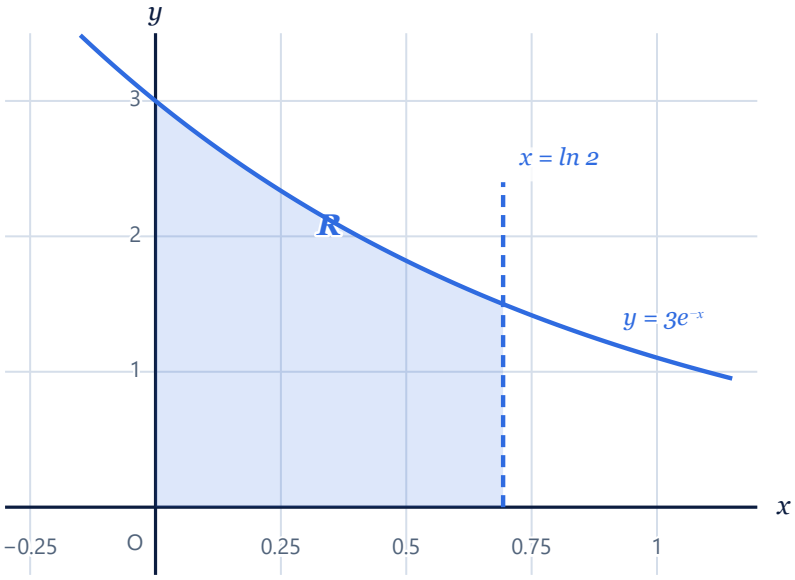
Further Maths topic practice

This topic online

1 The region R is bounded by the curve with equation $y = 3e^{-x}$, the coordinate axes and the line $x = \ln 2$, as shown in the diagram.

The region R is rotated through 2π radians about the x-axis.

Use calculus to find the exact volume of the solid generated.



(4 marks)

$V = \pi \int y^2 \, dx$

$y^2 = 9e^{-2x}$

Limits $x = 0$ to $x = \ln 2$

$V = \pi \int 9e^{-2x} \, dx$

$V = \pi \left[-\frac{9}{2} e^{-2x} \right]$

$e^{-2\ln 2} = e^{\ln(1/4)} = \frac{1}{4}$

$V = \pi \left(-\frac{9}{2} \times \frac{1}{4} + \frac{9}{2} \right)$

$V = \pi \left(-\frac{9}{8} + \frac{36}{8} \right)$

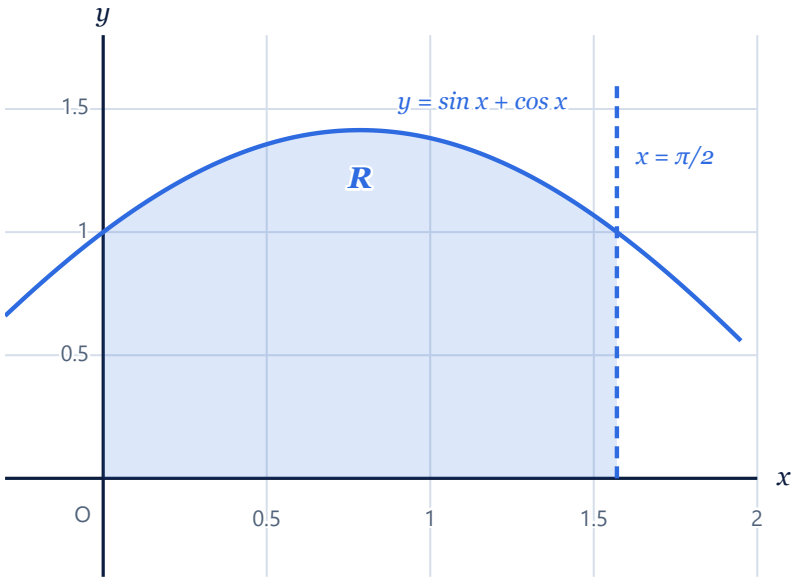
$$V = \frac{27\pi}{8}$$

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2 The region R is bounded by the curve with equation $y = \sin x + \cos x$, the coordinate axes and the line $x = \frac{\pi}{2}$

The region R is rotated through 2π radians about the x-axis.

Show that the volume of the solid generated is $\frac{\pi}{2}(\pi + 2)$



(4 marks)

$V = \pi\int y^2\,dx$

$y^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x$

$\sin^2 x + \cos^2 x = 1$ and $2\sin x\cos x = \sin 2x$

$y^2 = 1 + \sin 2x$

Limits $x = 0$ to $x = \frac{\pi}{2}$

$V = \pi\int(1 + \sin 2x)\,dx$

$V = \pi\left[x - \frac{1}{2}\cos 2x\right]$

$V = \pi\left(\left(\frac{\pi}{2} - \frac{1}{2}\cos\pi\right) - \left(0 - \frac{1}{2}\cos 0\right)\right)$

$V = \pi\left(\frac{\pi}{2} + \frac{1}{2} + \frac{1}{2}\right)$

$V = \pi\left(\frac{\pi}{2} + 1\right)$

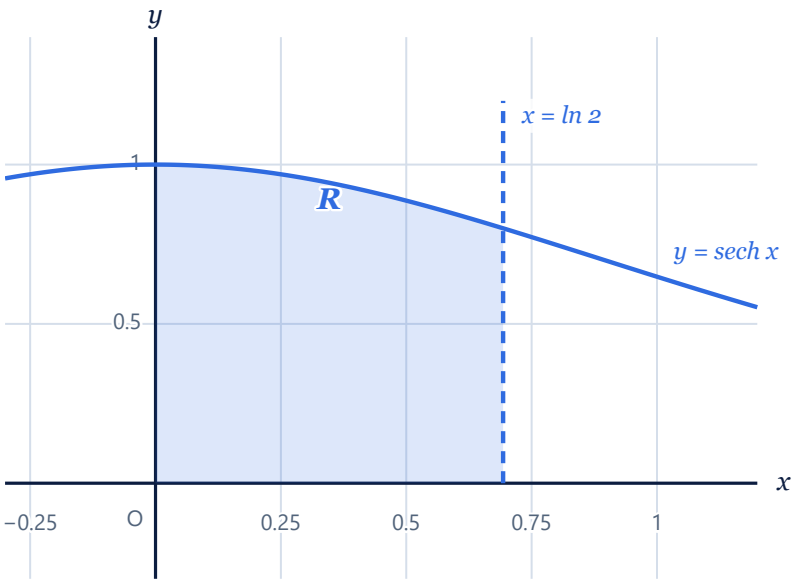
$\boxed{V = \frac{\pi}{2}(\pi + 2)\text{ (shown)}}$

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3 The region R is bounded by the curve with equation $y = \text{sech } x$, the coordinate axes and the line $x = \ln 2$

The region R is rotated through 360° about the x-axis.

Find the exact volume of the solid generated, giving your answer as a rational multiple of π.



(4 marks)

$V = \pi \int y^2 \, dx$

$V = \pi \int \text{sech}^2 x \, dx$

Limits $x = 0$ to $x = \ln 2$

$V = \pi [\tanh x]$

$V = \pi (\tanh(\ln 2) - \tanh 0)$

$\tanh x = \frac{(e^{2x} - 1)}{(e^{2x} + 1)}$ and $e^{2\ln 2} = 4$

$\tanh(\ln 2) = \frac{(4 - 1)}{(4 + 1)} = \frac{3}{5}$

$$V = \frac{3\pi}{5}$$

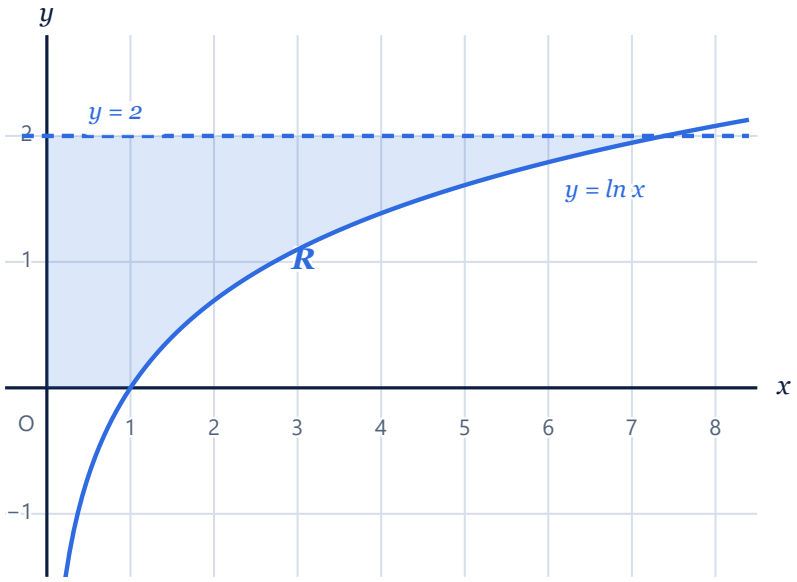
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4 The diagram shows part of the curve with equation $y = \ln x$

The region R, shown shaded, is bounded by the curve, the x-axis, the y-axis and the line with equation $y = 2$

The region R is rotated through 2π radians about the y-axis.

Use algebraic integration to find the exact volume of the solid generated.



(5 marks)

About the y-axis use $V = \pi \int x^2 \, dy$

$y = \ln x$

$x = e^y$

$x^2 = e^{2y}$

Limits $y = 0$ to $y = 2$

$V = \pi \int e^{2y} \, dy$

$V = \pi \left[\frac{1}{2} e^{2y} \right]$

$V = \pi \left(\frac{1}{2} e^4 - \frac{1}{2} e^0 \right)$

$$V = \frac{\pi}{2}(e^4 - 1)$$

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5 The curve C has equation $y = \text{artanh}\, x$, $0 \leq x < 1$

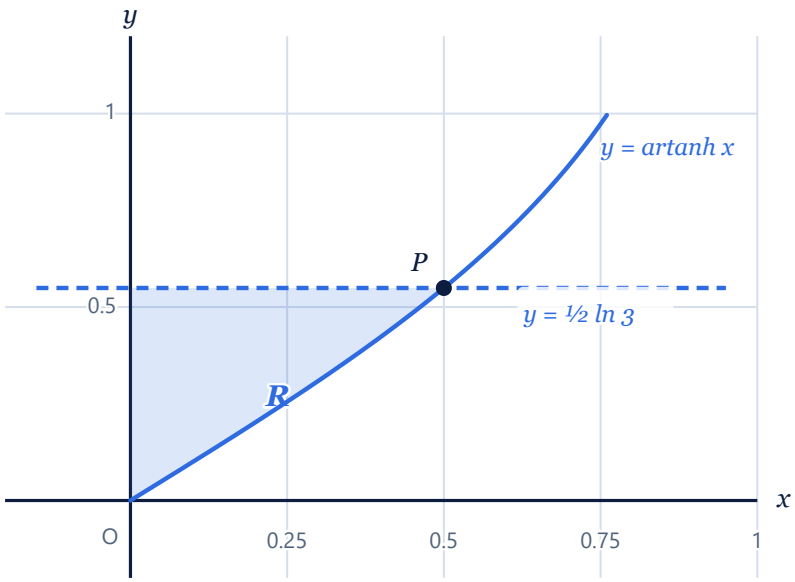
The line with equation $y = \frac{1}{2}\ln 3$ meets C at the point P.

a. Show that the x coordinate of P is $\frac{1}{2}$

The finite region R, shown shaded in the diagram, is bounded by C, the y-axis and the line with equation $y = \frac{1}{2}\ln 3$

The region R is rotated through 2π radians about the y-axis.

b. Use calculus to find the exact volume of the solid generated. Solutions based entirely on graphical or numerical methods are not acceptable.



(8 marks)

a)

Use the log form $\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{(1+x)}{(1-x)}\right)$

$$\text{artanh}\,\frac{1}{2} = \frac{1}{2}\ln\left(\frac{\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}\right)$$

$$\text{artanh}\,\frac{1}{2} = \frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right)$$

$$\text{artanh}\,\frac{1}{2} = \frac{1}{2}\ln 3$$

So at $y = \frac{1}{2}\ln 3$, $x = \frac{1}{2}$ (shown)

b)

About the y-axis use $V = \pi\int x^2\, dy$

$$y = \operatorname{artanh} x$$

$$x = \tanh y$$

$$x^2 = \tanh^2 y$$

Use $\tanh^2 y = 1 - \operatorname{sech}^2 y$

$$V = \pi \int (1 - \operatorname{sech}^2 y)\, dy$$

Limits $y = 0$ to $y = \frac{1}{2} \ln 3$

$$V = \pi [y - \tanh y]$$

From part (a), $\tanh(\frac{1}{2} \ln 3) = \frac{1}{2}$

$$V = \pi((\frac{1}{2} \ln 3 - \frac{1}{2}) - (0 - 0))$$

$$\boxed{V = \frac{\pi}{2}(\ln 3 - 1)}$$

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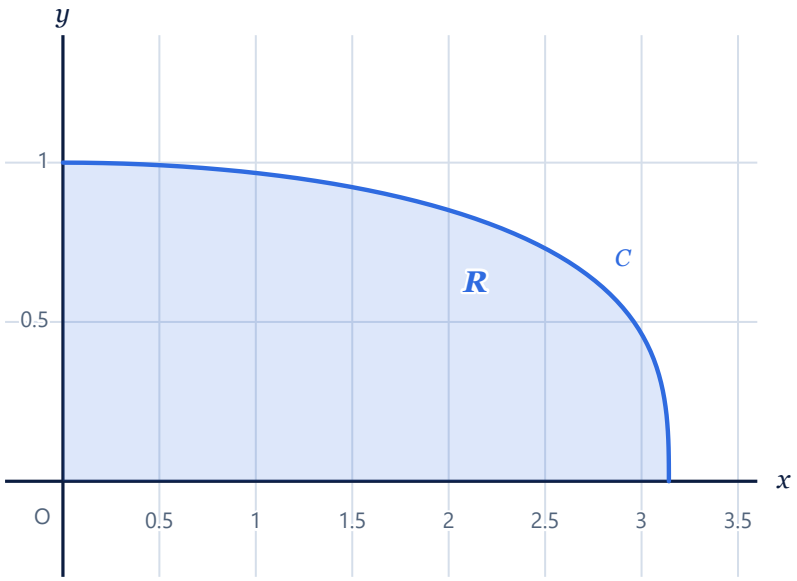
6 The curve C has parametric equations

$$x = 2t + \sin 2t, \quad y = \cos t, \quad 0 \leq t \leq \frac{\pi}{2}$$

The region R, shown shaded in the diagram, is bounded by C and the coordinate axes.

a. Show that $\frac{dx}{dt} = 4\cos^2 t$

b. The region R is rotated through 2π radians about the x-axis. Use algebraic integration to find the exact volume of the solid generated.



(7 marks)

a)

$$\frac{dx}{dt} = 2 + 2\cos 2t$$

$\cos 2t = 2\cos^2 t - 1$

$$\frac{dx}{dt} = 2 + 4\cos^2 t - 2$$

$$\boxed{\frac{dx}{dt} = 4\cos^2 t \text{ (shown)}}$$

b)

$$V = \pi \int y^2 \frac{dx}{dt}\, dt$$

$x = 0$ when $t = 0$ and $y = 0$ when $t = \frac{\pi}{2}$

$$V = \pi \int \cos^2 t \times 4\cos^2 t \, dt$$

$$V = 4\pi \int \cos^4 t \, dt$$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, so square it

$$\cos^4 t = \frac{1}{4}(1 + 2\cos 2t + \cos^2 2t)$$

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$$\cos^4 t = \frac{1}{4}\left(1 + 2\cos 2t + \frac{1}{2} + \frac{1}{2}\cos 4t\right)$$

$$\cos^4 t = \frac{1}{8}\left(3 + 4\cos 2t + \cos 4t\right)$$

$$V = \frac{\pi}{2}\int \left(3 + 4\cos 2t + \cos 4t\right) dt$$

$$V = \frac{\pi}{2}\left[3t + 2\sin 2t + \frac{1}{4}\sin 4t\right]$$

Limits $t = 0$ to $t = \frac{\pi}{2}$: $\sin \pi = \sin 2\pi = 0$

$$V = \frac{\pi}{2}\left(\frac{3\pi}{2} + 0 + 0\right)$$

$$\boxed{V = \frac{3\pi^2}{4}}$$

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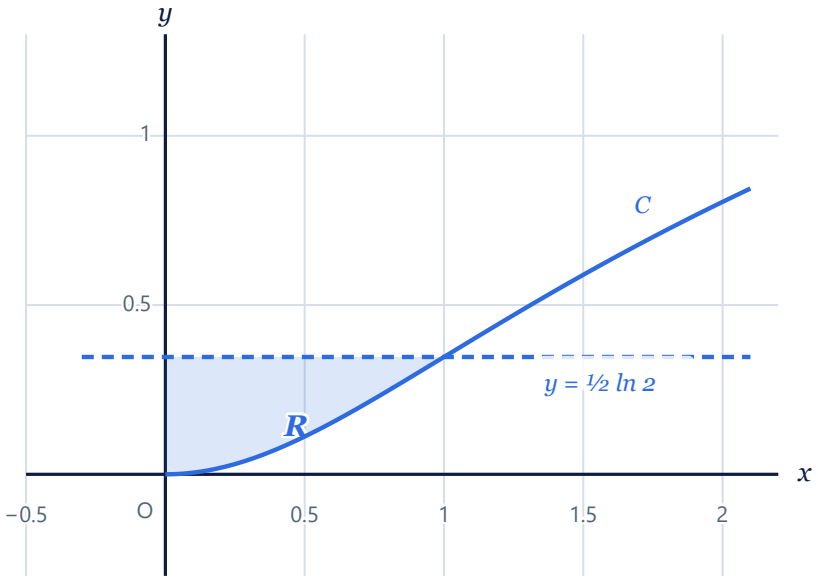
7 The curve C has parametric equations

$$x = \tan t, \quad y = \ln(\sec t), \quad 0 \leq t < \frac{\pi}{2}$$

a. Show that a cartesian equation of C is $y = \frac{1}{2}\ln(1 + x^2)$

The finite region R is bounded by C, the y-axis and the line with equation $y = \frac{1}{2}\ln 2$. The region R is rotated through 2π radians about the y-axis.

b. Use the parametric equations and algebraic integration to show that the volume of the solid generated is $\frac{\pi}{2}(1 - \ln 2)$



(7 marks)

a)

Use $\sec^2 t = 1 + \tan^2 t$

$$\sec^2 t = 1 + x^2$$

$$y = \ln(\sec t) = \frac{1}{2}\ln(\sec^2 t)$$

$$\boxed{y = \frac{1}{2}\ln(1 + x^2) \text{ (shown)}}$$

b)

About the y-axis $V = \pi\int x^2 \frac{dy}{dt}\, dt$

$$\frac{dy}{dt} = \frac{\sec t \tan t}{\sec t} = \tan t$$

Limits: $y = 0$ when $t = 0$

$$\frac{1}{2}\ln 2 = \ln(\sec t)$$

$$\sec t = \sqrt{2}$$

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$$t = \frac{\pi}{4}$$

$$V = \pi \int \tan^2 t \times \tan t \, dt$$

$$V = \pi \int (\sec^2 t - 1) \tan t \, dt$$

$$V = \pi \int (\sec^2 t \tan t - \tan t) \, dt$$

$$V = \pi \left[\frac{1}{2} \tan^2 t - \ln(\sec t) \right]$$

Limits $t = 0$ to $t = \frac{\pi}{4}$

$$V = \pi \left(\left(\frac{1}{2} \times 1 - \ln \sqrt{2} \right) - (0 - \ln 1) \right)$$

$$V = \pi \left(\frac{1}{2} - \frac{1}{2} \ln 2 \right)$$

$$\boxed{V = \frac{\pi}{2}(1 - \ln 2) \text{ (shown)}}$$

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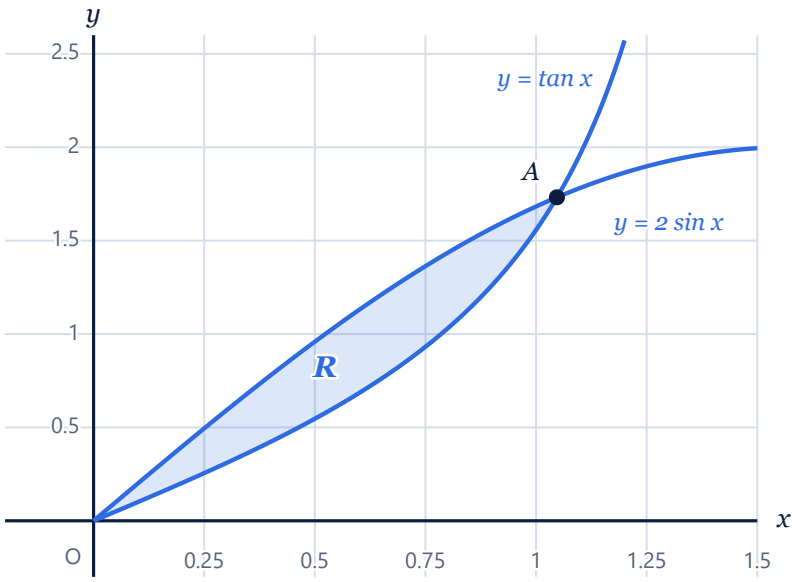
8 The diagram shows the curves with equations $y = 2 \sin x$ and $y = \tan x$, $0 \leq x < \frac{\pi}{2}$

The curves meet at the origin and at the point A.

a. Find the exact coordinates of A.

The finite region R, shown shaded, lies between the two curves. The region R is rotated through 2π radians about the x-axis.

b. Use algebraic integration to find the exact volume of the solid generated, giving your answer in the form $a\pi^2 + b\pi\sqrt{3}$, where a and b are rational numbers.



(7 marks)

a)

$$2\sin x = \frac{\sin x}{\cos x}$$

$$2\sin x \cos x - \sin x = 0$$

$$\sin x(2\cos x - 1) = 0$$

$\sin x = 0$ gives the origin

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

$$y = 2\sin\frac{\pi}{3} = \sqrt{3}$$

$$A\left(\frac{\pi}{3}, \sqrt{3}\right)$$

b)

The outer curve is $y = 2\sin x$, the inner is $y = \tan x$

$$V = \pi\int(4\sin^2 x - \tan^2 x)\,dx$$

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$4\sin^2 x = 2 - 2\cos 2x$ and $\tan^2 x = \sec^2 x - 1$

$$V = \pi\int (2 - 2\cos 2x - \sec^2 x + 1)\, dx$$

$$V = \pi\int (3 - 2\cos 2x - \sec^2 x)\, dx$$

$$V = \pi[3x - \sin 2x - \tan x]$$

Limits $x = 0$ to $x = \frac{\pi}{3}$

$$V = \pi(\pi - \sin\frac{2\pi}{3} - \tan\frac{\pi}{3})$$

$$V = \pi(\pi - \frac{\sqrt{3}}{2} - \sqrt{3})$$

$$V = \pi(\pi - \frac{3\sqrt{3}}{2})$$

$$\boxed{V = \pi^2 - \frac{3}{2}\pi\sqrt{3}}$$

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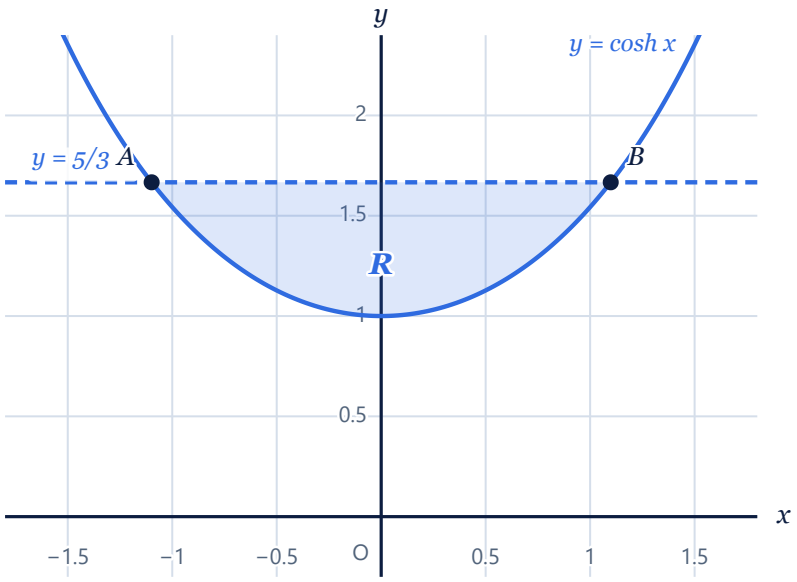
9 The diagram shows the curve with equation $y = \cosh x$ and the line with equation $y = \frac{5}{3}$

The line meets the curve at the points A and B.

a. Show that the x coordinates of A and B are $-\ln 3$ and $\ln 3$

The finite region R, shown shaded, is bounded by the curve and the line. The region R is rotated through 2π radians about the x-axis.

b. Use algebraic integration to show that the volume of the solid generated is $\frac{\pi}{9}(41 \ln 3 - 20)$



(8 marks)

a)

$$\cosh x = \frac{5}{3}$$

$$\frac{1}{2}(e^x + e^{-x}) = \frac{5}{3}$$

$$3e^x + 3e^{-x} = 10$$

Multiply by e^x

$$3e^{2x} - 10e^x + 3 = 0$$

$$(3e^x - 1)(e^x - 3) = 0$$

$$e^x = 3 \text{ or } e^x = \frac{1}{3}$$

$$\boxed{x = \ln 3 \text{ or } x = -\ln 3 \text{ (shown)}}$$

b)

Cylinder of radius $\frac{5}{3}$ minus the solid under $y = \cosh x$

$$V = \pi\int\left(\left(\frac{5}{3}\right)^2 - \cosh^2 x\right) dx$$

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$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$

$$V = \pi\int\left(\frac{25}{9} - \frac{1}{2} - \frac{1}{2}\cosh 2x\right)dx$$

$$V = \pi\int\left(\frac{41}{18} - \frac{1}{2}\cosh 2x\right)dx$$

The region is symmetrical, so use 2 x (0 to ln 3)

$$V = 2\pi\left[\frac{41}{18}x - \frac{1}{4}\sinh 2x\right]$$

$$\sinh(2\ln 3) = \frac{1}{2}\left(9 - \frac{1}{9}\right) = \frac{40}{9}$$

$$V = 2\pi\left(\frac{41}{18}\ln 3 - \frac{1}{4} \times \frac{40}{9}\right)$$

$$V = 2\pi\left(\frac{41}{18}\ln 3 - \frac{10}{9}\right)$$

$$V = \frac{41\pi}{9}\ln 3 - \frac{20\pi}{9}$$

$$\boxed{V = \frac{\pi}{9}(41\ln 3 - 20) \text{ (shown)}}$$

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10 a. Given that $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to show that

$$8\sin^4\theta \equiv \cos 4\theta - 4\cos 2\theta + 3$$

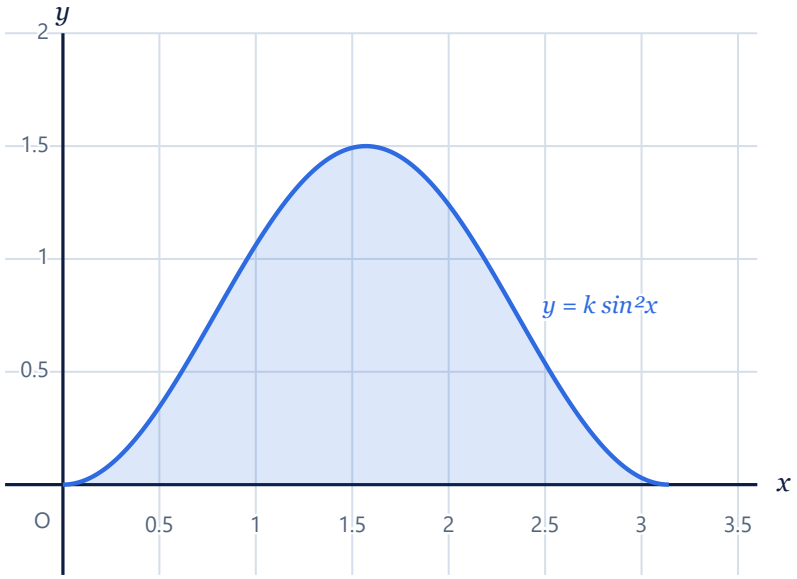
A wooden bead is modelled as the solid formed when the region bounded by the curve with equation $y = k\sin^2 x$, $0 \le x \le \pi$, and the x-axis is rotated through 2π radians about the x-axis, where k is a positive constant and the units are centimetres.

The widest part of the bead has a diameter of 3 cm.

b. Write down the value of k.

c. Using algebraic integration and the result in part (a), find the volume of the bead according to the model. Give your answer in cm^3 to 3 significant figures.

d. The real bead has a narrow hole drilled along its axis so that it can be threaded. State, with a reason, whether the model overestimates or underestimates the volume of wood in the bead.



(11 marks)

a)

$$z^n - z^{-n} = 2i\sin n\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$\left(z - \frac{1}{z}\right)^4 = (2i\sin\theta)^4 = 16\sin^4\theta$$

Expand with the binomial theorem

$$\left(z - \frac{1}{z}\right)^4 = z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$

$$\left(z - \frac{1}{z}\right)^4 = (z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6$$

$z^n + z^{-n} = 2\cos n\theta$

$$16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$$

$$\boxed{8\sin^4\theta \equiv \cos 4\theta - 4\cos 2\theta + 3 \text{ (shown)}}$$

b)

Greatest y is k (at $x = \frac{\pi}{2}$), the radius of the widest part

$2k = 3$

$\boxed{k = 1.5}$

c)

$V = \pi \int y^2 \, dx$

$V = \pi \int 2.25 \sin^4 x \, dx$

$V = \frac{2.25\pi}{8} \int (\cos 4x - 4\cos 2x + 3) \, dx$

$V = \frac{2.25\pi}{8} \left[\frac{1}{4} \sin 4x - 2\sin 2x + 3x \right]$

Limits $x = 0$ to $x = \pi$: $\sin 4\pi = \sin 2\pi = 0$

$V = \frac{2.25\pi}{8} \times 3\pi$

$V = \frac{27\pi^2}{32}$

$V = 8.327...$

$\boxed{V = 8.33 \text{ cm}^3}$

d)

The hole contains no wood but is part of the modelled solid

$\boxed{\text{The model overestimates the volume of wood}}$

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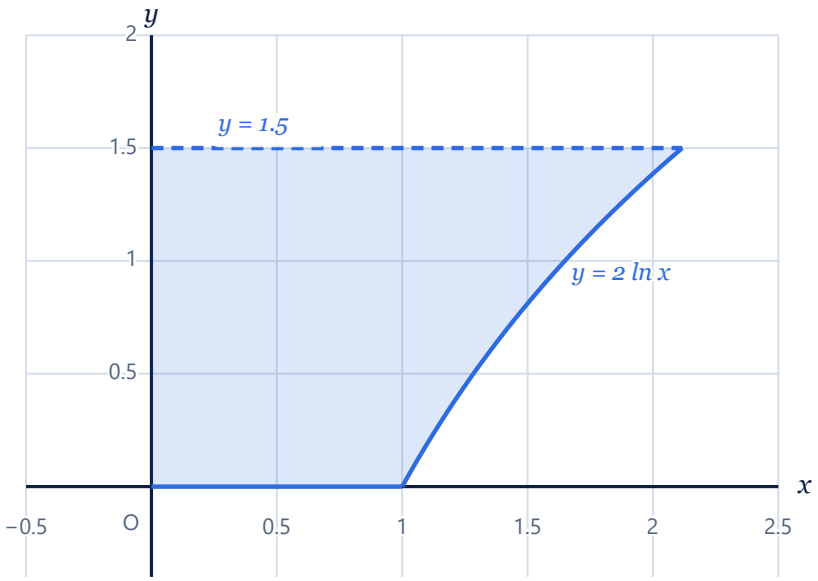
11 The inside of a bowl is modelled by rotating the curve with equation $y = 2 \ln x$, $1 \leq x \leq e^{0.75}$, through 2π radians about the y-axis, together with a flat circular base in the plane $y = 0$. The units are decimetres, and the bowl is 1.5 dm deep. (1 dm^3 = 1 litre.)

a. Show that, according to the model, the volume of water in the bowl when it is filled to a depth of h dm is $\pi(e^h - 1)$ dm^3

b. Find the capacity of the bowl in litres, to 3 significant figures.

c. Water is poured into the bowl at a constant rate of 0.5 litres per second. Find the rate at which the depth of the water is increasing at the instant when the depth is 0.8 dm.

d. Find the depth of the water when the bowl is exactly half full.



(10 marks)

a)

$y = 2\ln x$

$x = e^{y/2}$

$x^2 = e^y$

About the y-axis, from $y = 0$ to $y = h$

$V = \pi\int e^y \, dy$

$V = \pi[e^y]$

$V = \pi(e^h - e^0)$

$\boxed{V = \pi(e^h - 1) \text{ (shown)}}$

b)

$V = \pi(e^{1.5} - 1)$

$V = 10.938...$

$\boxed{10.9 \text{ Litres}}$

c)

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Chain rule: $\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh}$

$$\frac{dV}{dh} = \pi e^h$$

$$\frac{dh}{dt} = \frac{0.5}{\pi e^h}$$

$$\frac{dh}{dt} = \frac{0.5}{\pi e^{0.8}}$$

$$\frac{dh}{dt} = 0.07151...$$

0.0715 dm per second

d)

$$\pi(e^h - 1) = \frac{1}{2}\pi(e^{1.5} - 1)$$

$$e^h - 1 = \frac{1}{2}(e^{1.5} - 1)$$

$$e^h = \frac{(e^{1.5} + 1)}{2}$$

$$h = \ln\left(\frac{(e^{1.5} + 1)}{2}\right)$$

$$h = 1.008...$$

h = 1.01 dm

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12 A glass vase is 15 cm tall. Its base and its opening are both circles of diameter 8 cm. The curved side of the vase is modelled by rotating the curve C through 2π radians about the y-axis, where C has parametric equations

$$x = a \sin t, \quad y = -b \cos t, \quad \frac{\pi}{6} \leq t \leq \frac{5\pi}{6}$$

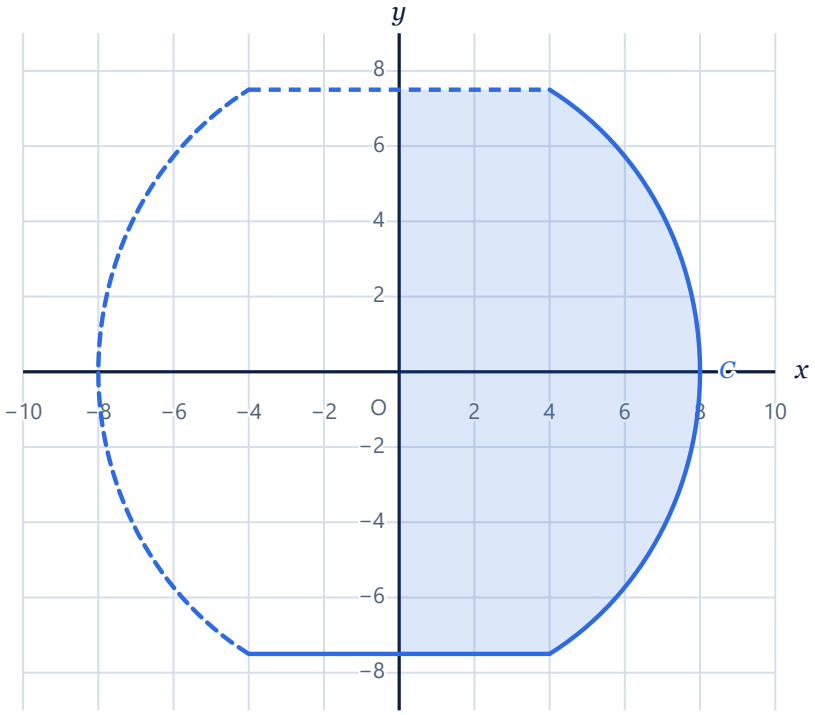
and a and b are positive constants. The origin O is on the axis of the vase, halfway up, and the units are centimetres. The thickness of the glass may be ignored.

a. Use the measurements to show that $a = 8$ and $b = 5\sqrt{3}$

b. Using algebraic integration, show that the capacity of the vase, according to the model, is 720π cm^3

c. The empty vase is filled from a tap at a constant rate of 40 cm^3 per second. Find, to the nearest second, how long it takes to fill the vase.

d. State one limitation of the model.



(10 marks)

a)

Radius of the base at $t = \frac{\pi}{6}$ is 4

$$a \sin \frac{\pi}{6} = 4$$

$$\frac{a}{2} = 4$$

$$\boxed{a = 8}$$

Height = y at $t = \frac{5\pi}{6}$ minus y at $t = \frac{\pi}{6}$

$$-b \cos \frac{5\pi}{6} - \left(-b \cos \frac{\pi}{6}\right) = 15$$

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$$\frac{b\sqrt{3}}{2} + \frac{b\sqrt{3}}{2} = 15$$

$$b\sqrt{3} = 15$$

$$b = \frac{15}{\sqrt{3}}$$

$\boxed{b = 5\sqrt{3} \text{ (shown)}}$

b)

About the y-axis $V = \pi\int x^2 \frac{dy}{dt}\,dt$

$$\frac{dy}{dt} = 5\sqrt{3}\sin t$$

$$V = \pi\int 64\sin^2 t \times 5\sqrt{3}\sin t\,dt$$

$$V = 320\sqrt{3}\pi\int \sin^3 t\,dt$$

$\sin^3 t = \sin t(1 - \cos^2 t)$

$$\int \sin^3 t\,dt = -\cos t + \frac{1}{3}\cos^3 t$$

Limits $t = \frac{\pi}{6}$ to $t = \frac{5\pi}{6}$

$$\text{At } \frac{5\pi}{6}: \frac{\sqrt{3}}{2} - \frac{1}{3} \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{8}$$

$$\text{At } \frac{\pi}{6}: -\frac{\sqrt{3}}{2} + \frac{1}{3} \times \frac{3\sqrt{3}}{8} = -\frac{3\sqrt{3}}{8}$$

$$\int \sin^3 t\,dt = \frac{3\sqrt{3}}{8} + \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}$$

$$V = 320\sqrt{3}\pi \times \frac{3\sqrt{3}}{4}$$

$$V = 80 \times 9\pi$$

$\boxed{V = 720\pi \text{ cm}^3 \text{ (shown)}}$

c)

$$\text{time} = 720\pi \div 40$$

$$\text{time} = 18\pi = 56.54\ldots$$

$\boxed{57 \text{ seconds}}$

d)

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

The glass has a thickness, so the real capacity is less than the model gives

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