Algorithms and sorting

Worked solutions

Further Maths topic practice

This topic online

1 Use a bubble sort, starting at the left-hand end of the list, to sort these numbers into ascending order.

34 18 52 27 41 9 30

You should show the state of the list after each pass. (4 marks)

Pass 1: compare each adjacent pair, swap if out of order

34 and 18: swap

34 and 52: no swap

52 and 27: swap

52 and 41: swap

52 and 9: swap

52 and 30: swap

Pass 1: 18 34 27 41 9 30 52

Pass 2: 18 27 34 9 30 41 52

Pass 3: 18 27 9 30 34 41 52

Pass 4: 18 9 27 30 34 41 52

Pass 5: 9 18 27 30 34 41 52

Pass 6: 9 18 27 30 34 41 52

No swaps in pass 6, so the list is sorted

9 18 27 30 34 41 52

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2 A bubble sort is an algorithm of quadratic order. A laptop takes 0.8 seconds to bubble sort a list of 400 names.

a. Estimate the time it would take to bubble sort a list of 2000 names.

A different algorithm has cubic order. It takes 1.2 seconds when the input has size 50.

b. Estimate the time it would take when the input has size 200.

(4 marks)

a) Quadratic order: time $\propto n^2$

$$\text{scale factor} = \frac{2000}{400} = 5$$

$$\text{time} = 0.8 \times 5^2$$

$$\text{time} = 20 \text{ s}$$

20 seconds

b) Cubic order: time $\propto n^3$

$$\text{scale factor} = \frac{200}{50} = 4$$

$$\text{time} = 1.2 \times 4^3$$

$$\text{time} = 76.8 \text{ s}$$

76.8 seconds

3 Use the Euclidean algorithm to find the highest common factor of 4199 and 1547. Show each step of your working.

(3 marks)

Divide the larger number by the smaller, keep the remainder

$$4199 = 2 \times 1547 + 1105$$

$$1547 = 1 \times 1105 + 442$$

$$1105 = 2 \times 442 + 221$$

$$442 = 2 \times 221 + 0$$

Remainder 0: the HCF is the last divisor

HCF = 221

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4 Use a quick sort to sort the list below into ascending order. You should show the result of each pass and identify your pivots clearly.

47 12 85 33 60 21 74 9 56 38

(4 marks)

Pivot = middle item (right of the middle if the sublist has an even number of items)

Pass 1: pivot 21

12 9 [21] 47 85 33 60 74 56 38

Pass 2: pivots 9, 60

[9] 12 [21] 47 33 56 38 [60] 85 74

Pass 3: pivots 56, 74

[9] 12 [21] 47 33 38 [56] [60] [74] 85

Pass 4: pivot 33

[9] 12 [21] [33] 47 38 [56] [60] [74] 85

Pass 5: pivot 38

[9] 12 [21] [33] [38] 47 [56] [60] [74] 85

Every sublist now has one item, so the sort is complete

9 12 21 33 38 47 56 60 74 85

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5 The names of nine members of a chess club are listed below.

Priya Callum Zara Mohammed Erin Tomasz Hannah Kofi Beth

Use a quick sort to put the names into alphabetical order. You should show the result of each pass and identify your pivots clearly.

(4 marks)

Pivot = middle item (right of the middle if the sublist has an even number of items)

Pass 1: pivot Erin

Callum Beth [Erin] Priya Zara Mohammed Tomasz Hannah Kofi

Pass 2: pivots Beth, Tomasz

[Beth] Callum [Erin] Priya Mohammed Hannah Kofi [Tomasz] Zara

Pass 3: pivot Hannah

[Beth] Callum [Erin] [Hannah] Priya Mohammed Kofi [Tomasz] Zara

Pass 4: pivot Mohammed

[Beth] Callum [Erin] [Hannah] Kofi [Mohammed] Priya [Tomasz] Zara

Every sublist now has one item, so the sort is complete

Beth, Callum, Erin, Hannah, Kofi, Mohammed, Priya, Tomasz, Zara

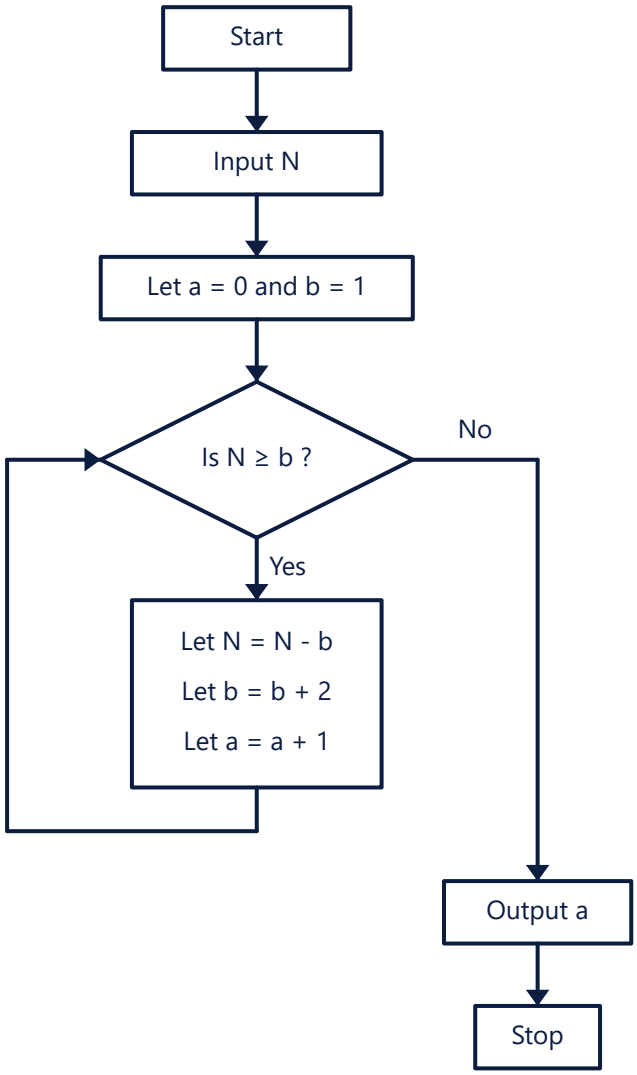
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6 The flow chart describes an algorithm.

a. Trace the algorithm with input $N = 40$, recording the values of N, a and b each time round the loop. State the output.

b. Write down the output when the input is $N = 90$

c. State what the algorithm finds.



(6 marks)

a) Trace each time round the loop

Start: $N = 40$, $a = 0$, $b = 1$

$40 \geq 1$? Yes: $N = 39$, $b = 3$, $a = 1$

$39 \geq 3$? Yes: $N = 36$, $b = 5$, $a = 2$

$36 \geq 5$? Yes: $N = 31$, $b = 7$, $a = 3$

$31 \geq 7$? Yes: $N = 24$, $b = 9$, $a = 4$

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$24 \geqslant 9$? Yes: $N = 15$, $b = 11$, $a = 5$

$15 \geqslant 11$? Yes: $N = 4$, $b = 13$, $a = 6$

$4 \geqslant 13$? No: output $a = 6$

Output 6

b) $N = 90$: subtract 1, 3, 5, 7, ...

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81$

After 9 loops $N = 9$ and $b = 19$, and $9 < 19$

Output 9

c) Sum of the first k odd numbers $= k^2$

a counts how many odd numbers fit into N

The integer part of $\sqrt{N}$

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7 The list below is to be sorted into descending order using a bubble sort, starting at the left-hand end of the list.

15 42 8 27 33 19

a. Carry out the bubble sort, showing the list after each pass.

b. Find the total number of comparisons and the total number of swaps made.

c. Find, in terms of n, the maximum number of comparisons needed to bubble sort a list of n numbers.

(7 marks)

a)

Descending: swap when the left item is smaller

Pass 1: 42 15 27 33 19 8 (5 comparisons, 4 swaps)

Pass 2: 42 27 33 19 15 8 (4 comparisons, 3 swaps)

Pass 3: 42 33 27 19 15 8 (3 comparisons, 1 swap)

Pass 4: 42 33 27 19 15 8 (2 comparisons, 0 swaps)

No swaps in pass 4: sorted

42 33 27 19 15 8

b) Add up the counts

comparisons = $5 + 4 + 3 + 2 = 14$

swaps = $4 + 3 + 1 + 0 = 8$

14 comparisons, 8 swaps

c) Worst case: $n - 1$ passes, pass k makes $n - k$ comparisons

total = $(n - 1) + (n - 2) + \ldots + 2 + 1$

total = $\frac{n(n - 1)}{2}$

$\frac{1}{2}n(n - 1)$

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8 On average a quick sort has order n log n. A computer takes 0.36 seconds to quick sort a list of 1000 numbers.

a. Estimate how long it would take to quick sort a list of 8000 numbers.

An algorithm of cubic order takes 0.05 seconds for an input of size 20.

b. Find the largest input size that the computer can process with this algorithm in one minute. (5 marks)

a) Time $\propto n \log n$ (any base)

$$\text{time} = 0.36 \times \frac{8000 \log 8000}{1000 \log 1000}$$

$$\text{time} = 0.36 \times 8 \times \frac{3.9031}{3}$$

$$\text{time} = 3.75 \text{ s}$$

3.75 seconds

b) Time $\propto n^3$

$$0.05 \times \left(\frac{n}{20}\right)^3 = 60$$

$$\left(\frac{n}{20}\right)^3 = 1200$$

$$n^3 = 1200 \times 8000 = 9\,600\,000$$

$$n = 212.53$$

Round down: it must finish within the minute

$n = 212$

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9 An algorithm is shown below. INT(x) means the integer part of x.

Step 1 Input a positive integer N
Step 2 Let $P = 1$ and $B = 0$
Step 3 Let R be the remainder when N is divided by 2
Step 4 Let $B = B + R \times P$
Step 5 Let $N = \text{INT}(N \div 2)$
Step 6 Let $P = 10P$
Step 7 If $N > 0$ go to Step 3
Step 8 Output B
Step 9 Stop

a. Trace the algorithm for $N = 45$, showing the values of R, B, N and P each time Step 6 is completed. State the output.

b. Explain what the algorithm does.

c. Find how many times Step 3 is carried out when $N = 1000$

(6 marks)

a) Start: N = 45, P = 1, B = 0

Loop 1: R = 1, B = 1, N = 22, P = 10

Loop 2: R = 0, B = 1, N = 11, P = 100

Loop 3: R = 1, B = 101, N = 5, P = 1000

Loop 4: R = 1, B = 1101, N = 2, P = 10000

Loop 5: R = 0, B = 1101, N = 1, P = 100000

Loop 6: R = 1, B = 101101, N = 0, P = 1000000

N = 0 so the loop stops

Output B = 101101

b) Check

101101 in base 2 = 32 + 8 + 4 + 1 = 45

It writes N in binary

c) Number of passes = number of binary digits of N

$1000 < 1024 = 2^{10}$, and $1000 \geq 512 = 2^9$

10 times

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10 The six distinct integers 24, x, 17, 31, 12, 28 are to be sorted into ascending order using a bubble sort, starting at the left-hand end of the list.

After the first pass the list is 24 17 x 12 28 31

a. Explain why $24 < x < 31$

After the second pass the list is 17 24 12 28 x 31

b. Find the possible values of x.

c. Taking the smaller value of x, complete the sort, showing the list after each further pass. (6 marks)

a) Pass 1 starts 24 v x

24 stays first, so no swap: $x > 24$

x swaps with 17, then x is compared with 31

x is still before 31, so no swap: $x < 31$

$\boxed{24 < x < 31}$

b) Pass 2: 17 24 x 12 28 31

x has moved past 12 and past 28

so $x > 28$

Combine with $24 < x < 31$, x an integer, $x \neq 31$

$\boxed{x = 29 \text{ or } x = 30}$

c) $x = 29$, continue from 17 24 12 28 29 31

Pass 3: 17 12 24 28 29 31

Pass 4: 12 17 24 28 29 31

Pass 5: 12 17 24 28 29 31 (no swaps: sorted)

$\boxed{12\ 17\ 24\ 28\ 29\ 31}$

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11 The list below is to be sorted into descending order.

63 18 45 92 27 71 54 36

a. Use a quick sort to sort the list. You should show the result of each pass and identify your pivots clearly.

b. Find the total number of comparisons made in your quick sort.

c. A bubble sort, starting at the left-hand end, is used on the original list instead. Find the number of passes and the total number of comparisons it makes, and state which method was quicker for this list.

(8 marks)

a) Descending: bigger items go left of the pivot

Pivot = middle item (right of the middle if the sublist has an even number of items)

Pass 1: pivot 27

63 45 92 71 54 36 [27] 18

Pass 2: pivot 71

92 [71] 63 45 54 36 [27] 18

Pass 3: pivot 54

92 [71] 63 [54] 45 36 [27] 18

Pass 4: pivot 36

92 [71] 63 [54] 45 [36] [27] 18

Every sublist now has one item, so the sort is complete

92 71 63 54 45 36 27 18

b) A sublist of m items makes m - 1 comparisons with its pivot

Pass 1: 7 comparisons

Pass 2: 5 comparisons

Pass 3: 3 comparisons

Pass 4: 1 comparison

total = 7 + 5 + 3 + 1 = 16

16 comparisons

c) Bubble sort, descending

Pass 1: 63 45 92 27 71 54 36 18 (7 comparisons)

Pass 2: 63 92 45 71 54 36 27 18 (6 comparisons)

Pass 3: 92 63 71 54 45 36 27 18 (5 comparisons)

Pass 4: 92 71 63 54 45 36 27 18 (4 comparisons)

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Pass 5: 92 71 63 54 45 36 27 18 (3 comparisons, no swaps)

total = 7 + 6 + 5 + 4 + 3 = 25

5 passes, 25 comparisons: quick sort quicker

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12 An algorithm is shown below.

Step 1 Input a positive integer n
Step 2 Let $i = 1$ and $T = 0$
Step 3 Let $j = i$
Step 4 Let $T = T + j$
Step 5 Let $j = j + 1$
Step 6 If $j \leq n$ go to Step 4
Step 7 Let $i = i + 1$
Step 8 If $i \leq n$ go to Step 3
Step 9 Output T
Step 10 Stop

a. Trace the algorithm with $n = 3$, showing the values of i, j and T each time Step 4 is completed. State the output.

b. Explain why the output is $1^2 + 2^2 + \ldots + n^2$ and hence write down the output in terms of n.

c. Show that Step 4 is carried out $\frac{1}{2}n(n + 1)$ times and state the order of the algorithm.

d. The algorithm takes 0.004 seconds when $n = 500$. Estimate the time it takes when $n = 3000$

(9 marks)

a) Each time Step 4 is completed

$i = 1, j = 1, T = 1$

$i = 1, j = 2, T = 3$

$i = 1, j = 3, T = 6$

$i = 2, j = 2, T = 8$

$i = 2, j = 3, T = 11$

$i = 3, j = 3, T = 14$

i becomes $4 > 3$, so stop

Output $T = 14$

b) Count how many times each j is added

j is added once for each $i = 1, 2, \ldots, j$

so j is added j times, giving $j \times j = j^2$

$T = 1^2 + 2^2 + \ldots + n^2$

$T = \frac{1}{6}n(n + 1)(2n + 1)$

Check $n = 3$: $\frac{1}{6} \times 3 \times 4 \times 7 = 14$

$\frac{1}{6}n(n + 1)(2n + 1)$

c) For each i, j runs from i to n

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number of times $= n + (n-1) + \ldots + 2 + 1$

number of times $= \frac{1}{2}n(n+1)$

Highest power of n is 2

Quadratic order

d) Time $\propto n^2$

$$\text{time} = 0.004 \times \left(\frac{3000}{500}\right)^2$$

$$\text{time} = 0.004 \times 36$$

$$\text{time} = 0.144 \text{ s}$$

0.144 seconds

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