Bin packing

Worked solutions

Further Maths topic practice

This topic online

1 Boxes with the following masses, in kilograms, are to be loaded onto trolleys. Each trolley can carry at most 20 kg.

8 11 6 14 3 9 12 5 7

Calculate a lower bound for the number of trolleys needed. You must make your method clear. (2 marks)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 8 + 11 + 6 + 14 + 3 + 9 + 12 + 5 + 7 = 75$$

$$\frac{75}{20} = 3.750$$

$$\text{Lower bound} = 4$$

Round up: you cannot use part of a trolley

4 trolleys

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2 Use the first-fit bin packing algorithm to pack the following numbers into bins of size 25

7 13 9 4 15 6 11 8 2

(3 marks)

Each number goes in the first bin it fits in

7: Bin 1

13: Bin 1

9: too big for Bin 1 (space 5), so Bin 2

4: Bin 1

15: too big for Bin 1 (space 1), so Bin 2

6: too big for Bin 1 (space 1), Bin 2 (space 1), so Bin 3

11: too big for Bin 1 (space 1), Bin 2 (space 1), so Bin 3

8: too big for Bin 1 (space 1), Bin 2 (space 1), so Bin 3

2: too big for Bin 1 (space 1), Bin 2 (space 1), Bin 3 (space 0), so Bin 4

Final packing

Bin 1: 7 13 4 (total 24, space 1)

Bin 2: 9 15 (total 24, space 1)

Bin 3: 6 11 8 (total 25, space 0)

Bin 4: 2 (total 2, space 23)

4 bins

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3 Use the first-fit decreasing bin packing algorithm to pack the following numbers into bins of size 30

12 5 18 9 14 7 16 3

(3 marks)

First sort into descending order

18 16 14 12 9 7 5 3

Then first-fit

18: Bin 1

16: too big for Bin 1 (space 12), so Bin 2

14: too big for Bin 1 (space 12), so Bin 2

12: Bin 1

9: too big for Bin 1 (space 0), Bin 2 (space 0), so Bin 3

7: too big for Bin 1 (space 0), Bin 2 (space 0), so Bin 3

5: too big for Bin 1 (space 0), Bin 2 (space 0), so Bin 3

3: too big for Bin 1 (space 0), Bin 2 (space 0), so Bin 3

Final packing

Bin 1: 18 12 (total 30, space 0)

Bin 2: 16 14 (total 30, space 0)

Bin 3: 9 7 5 3 (total 24, space 6)

3 bins

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4 A dressmaker needs these lengths of ribbon, in metres. Ribbon is sold on reels of 5 m and each length must be cut from a single reel.

2.4 1.7 3.1 0.9 2.6 1.3 3.5 2.2 1.8

a. Calculate a lower bound for the number of reels needed.

b. Use the first-fit bin packing algorithm to show how the lengths could be cut from the reels.

c. Use the first-fit decreasing algorithm to show how the lengths could be cut from the reels.

d. Explain whether your answer to part (c) is optimal.

(7 marks)

a)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 2.4 + 1.7 + 3.1 + 0.9 + 2.6 + 1.3 + 3.5 + 2.2 + 1.8 = 19.5$$

$$\frac{19.5}{5} = 3.900$$

$$\text{Lower bound} = 4$$

4 reels

b) First-fit, in the order given

2.4: Bin 1

1.7: Bin 1

3.1: too big for Bin 1 (space 0.9), so Bin 2

0.9: Bin 1

2.6: too big for Bin 1 (space 0), Bin 2 (space 1.9), so Bin 3

1.3: too big for Bin 1 (space 0), so Bin 2

3.5: too big for Bin 1 (space 0), Bin 2 (space 0.6), Bin 3 (space 2.4), so Bin 4

2.2: too big for Bin 1 (space 0), Bin 2 (space 0.6), so Bin 3

1.8: too big for Bin 1 (space 0), Bin 2 (space 0.6), Bin 3 (space 0.2), Bin 4 (space 1.5), so Bin 5

Bin 1: 2.4 1.7 0.9 (total 5, space 0)

Bin 2: 3.1 1.3 (total 4.4, space 0.6)

Bin 3: 2.6 2.2 (total 4.8, space 0.2)

Bin 4: 3.5 (total 3.5, space 1.5)

Bin 5: 1.8 (total 1.8, space 3.2)

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5 reels

c) Sort into descending order

3.5 3.1 2.6 2.4 2.2 1.8 1.7 1.3 0.9

3.5: Bin 1

3.1: too big for Bin 1 (space 1.5), so Bin 2

2.6: too big for Bin 1 (space 1.5), Bin 2 (space 1.9), so Bin 3

2.4: too big for Bin 1 (space 1.5), Bin 2 (space 1.9), so Bin 3

2.2: too big for Bin 1 (space 1.5), Bin 2 (space 1.9), Bin 3 (space 0), so Bin 4

1.8: too big for Bin 1 (space 1.5), so Bin 2

1.7: too big for Bin 1 (space 1.5), Bin 2 (space 0.1), Bin 3 (space 0), so Bin 4

1.3: Bin 1

0.9: too big for Bin 1 (space 0.2), Bin 2 (space 0.1), Bin 3 (space 0), so Bin 4

Bin 1: 3.5 1.3 (total 4.8, space 0.2)

Bin 2: 3.1 1.8 (total 4.9, space 0.1)

Bin 3: 2.6 2.4 (total 5, space 0)

Bin 4: 2.2 1.7 0.9 (total 4.8, space 0.2)

4 reels

d) Compare with the lower bound

4 reels = lower bound 4

Optimal: fewer than the lower bound is impossible

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5 The numbers below are to be packed into bins of size 20

11 4 7 9 13 6 8 2 10 5

a. Use full-bin packing to pack the numbers into bins.

b. Explain why your packing is optimal.

(5 marks)

a) Find combinations that make exactly 20

$11 + 9 = 20$

$13 + 7 = 20$

$10 + 6 + 4 = 20$

Left over: 8, 2, 5

$8 + 2 + 5 = 15$

$15 \leq 20$, so these share one bin

Bin 1: 11 9

Bin 2: 13 7

Bin 3: 10 6 4

Bin 4: 8 2 5

4 bins

b)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 11 + 4 + 7 + 9 + 13 + 6 + 8 + 2 + 10 + 5 = 75$$

$$\frac{75}{20} = 3.750$$

$$\text{Lower bound} = 4$$

Packing uses 4 bins = Lower bound

Optimal

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6 The eight distinct integers 14, 9, x, 17, 6, 11, 8, 5 are packed, in that order, into bins of size 30 using the first-fit algorithm. The result is

Bin 1: 14 9 6
Bin 2: x 17
Bin 3: 11 8 5

a. Explain why $8 < x \leq 13$

b. Hence write down the possible values of x.

c. Given that $x = 12$, use the first-fit decreasing algorithm to pack the numbers into bins of size 30, and state whether your packing is optimal.

(7 marks)

a) Use each placement as an inequality

x not in Bin 1 (14 + 9 = 23 there):

$23 + x > 30$

$x > 7$

17 joins x in Bin 2:

$x + 17 \leq 30$

$x \leq 13$

5 does not fit in Bin 2:

$x + 17 + 5 > 30$

$x > 8$

$\boxed{8 < x \leq 13}$

b) Integers 9 to 13, all numbers distinct

9 and 11 are already in the list

$\boxed{x = 10, 12 \text{ or } 13}$

c) $x = 12$, descending order

17 14 12 11 9 8 6 5

17: Bin 1

14: too big for Bin 1 (space 13), so Bin 2

12: Bin 1

11: too big for Bin 1 (space 1), so Bin 2

9: too big for Bin 1 (space 1), Bin 2 (space 5), so Bin 3

8: too big for Bin 1 (space 1), Bin 2 (space 5), so Bin 3

6: too big for Bin 1 (space 1), Bin 2 (space 5), so Bin 3

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5: too big for Bin 1 (space 1), so Bin 2

Bin 1: 17 12 (total 29, space 1)

Bin 2: 14 11 5 (total 30, space 0)

Bin 3: 9 8 6 (total 23, space 7)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 14 + 9 + 12 + 17 + 6 + 11 + 8 + 5 = 82$$

$$\frac{82}{30} = 2.733$$

$$\text{Lower bound} = 3$$

3 bins = lower bound: optimal

7 The numbers 18, 7, 25, 12, 9, 20, 14, 6 are packed, in that order, into bins of size n, where n is an integer, using the first-fit algorithm. The result is

Bin 1: 18 7 12
Bin 2: 25 9
Bin 3: 20 14
Bin 4: 6

Find the possible values of n. You must give reasons for your answer. (4 marks)

Bins that were filled give $n \geqslant$ total

$18 + 7 + 12 = 37$, so $n \geqslant 37$

Items that did not fit give $n <$ total

9 not in Bin 1: $37 + 9 = 46 > n$

20 not in Bin 1 or 2: $34 + 20 = 54 > n$

14 not in Bin 2: $34 + 14 = 48 > n$

6 not in Bin 1: $37 + 6 = 43 > n$

6 not in Bin 2: $34 + 6 = 40 > n$

6 not in Bin 3: $20 + 14 + 6 = 40 > n$

The tightest conditions

$37 \leqslant n < 40$

$n = 37$, 38 or 39

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8 The eight distinct integers below are to be packed into bins of size 24

10 14 x 16 8 13 11 3

It is known that x is the largest number in the list and that $x < 24$

a. Explain why the numbers cannot be packed into 3 bins.

b. Given that the numbers can be packed into 4 bins, find the range of possible values of x.

c. Taking the largest possible value of x, use the first-fit algorithm to pack the numbers, and comment on the result.

(7 marks)

a) Total of the known numbers

$10 + 14 + 16 + 8 + 13 + 11 + 3 = 75$

x is the largest, so $x \geqslant 17$

total $\geqslant 75 + 17 = 92$

3 bins hold $3 \times 24 = 72$

$92 > 72$, so 3 bins are not enough

b) 4 bins hold $4 \times 24 = 96$

$75 + x \leqslant 96$

$x \leqslant 21$

x is the largest number: $x \geqslant 17$

Check $x = 21$ fits in 4 bins

$21 + 3$, $16 + 8$, $14 + 10$, $13 + 11$

A smaller x fits in the same bin as 3

$17 \leqslant x \leqslant 21$

c) $x = 21$, first-fit in the order given

10: Bin 1

14: Bin 1

21: too big for Bin 1 (space 0), so Bin 2

16: too big for Bin 1 (space 0), Bin 2 (space 3), so Bin 3

8: too big for Bin 1 (space 0), Bin 2 (space 3), so Bin 3

13: too big for Bin 1 (space 0), Bin 2 (space 3), Bin 3 (space 0), so Bin 4

11: too big for Bin 1 (space 0), Bin 2 (space 3), Bin 3 (space 0), so Bin 4

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3: too big for Bin 1 (space 0), so Bin 2

Bin 1: 10 14 (total 24, space 0)

Bin 2: 21 3 (total 24, space 0)

Bin 3: 16 8 (total 24, space 0)

Bin 4: 13 11 (total 24, space 0)

Part (a): at least 4 bins are needed

4 bins: optimal

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9 A carpenter needs to cut shelves of the following lengths, in metres, from planks that are 3.6 m long. Each shelf must be cut from a single plank.

1.2 0.8 2.1 1.5 0.6 1.9 1.1 0.7 1.4 2.4

a. Calculate a lower bound for the number of planks needed.

b. Use the first-fit bin packing algorithm to allocate the shelves to planks.

c. Use the first-fit decreasing algorithm to allocate the shelves to planks.

d. State which allocation the carpenter should use, and whether it is optimal. Give a reason. (8 marks)

a)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 1.2 + 0.8 + 2.1 + 1.5 + 0.6 + 1.9 + 1.1 + 0.7 + 1.4 + 2.4$$

$$= 13.7$$

$$\frac{13.7}{3.6} = 3.806$$

Lower bound = 4

4 planks

b) First-fit, order given

1.2: Bin 1

0.8: Bin 1

2.1: too big for Bin 1 (space 1.6), so Bin 2

1.5: Bin 1

0.6: too big for Bin 1 (space 0.1), so Bin 2

1.9: too big for Bin 1 (space 0.1), Bin 2 (space 0.9), so Bin 3

1.1: too big for Bin 1 (space 0.1), Bin 2 (space 0.9), so Bin 3

0.7: too big for Bin 1 (space 0.1), so Bin 2

1.4: too big for Bin 1 (space 0.1), Bin 2 (space 0.2), Bin 3 (space 0.6), so Bin 4

2.4: too big for Bin 1 (space 0.1), Bin 2 (space 0.2), Bin 3 (space 0.6), Bin 4 (space 2.2), so Bin 5

Bin 1: 1.2 0.8 1.5 (total 3.5, space 0.1)

Bin 2: 2.1 0.6 0.7 (total 3.4, space 0.2)

Bin 3: 1.9 1.1 (total 3, space 0.6)

Bin 4: 1.4 (total 1.4, space 2.2)

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Bin 5: 2.4 (total 2.4, space 1.2)

5 planks

c) Descending order

2.4 2.1 1.9 1.5 1.4 1.2 1.1 0.8 0.7 0.6

2.4: Bin 1

2.1: too big for Bin 1 (space 1.2), so Bin 2

1.9: too big for Bin 1 (space 1.2), Bin 2 (space 1.5), so Bin 3

1.5: too big for Bin 1 (space 1.2), so Bin 2

1.4: too big for Bin 1 (space 1.2), Bin 2 (space 0), so Bin 3

1.2: Bin 1

1.1: too big for Bin 1 (space 0), Bin 2 (space 0), Bin 3 (space 0.3), so Bin 4

0.8: too big for Bin 1 (space 0), Bin 2 (space 0), Bin 3 (space 0.3), so Bin 4

0.7: too big for Bin 1 (space 0), Bin 2 (space 0), Bin 3 (space 0.3), so Bin 4

0.6: too big for Bin 1 (space 0), Bin 2 (space 0), Bin 3 (space 0.3), so Bin 4

Bin 1: 2.4 1.2 (total 3.6, space 0)

Bin 2: 2.1 1.5 (total 3.6, space 0)

Bin 3: 1.9 1.4 (total 3.3, space 0.3)

Bin 4: 1.1 0.8 0.7 0.6 (total 3.2, space 0.4)

4 planks

d) Compare

First-fit: 5 planks, first-fit decreasing: 4 planks

Lower bound = 4

Use first-fit decreasing: optimal

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10 A courier has ten parcels to deliver. Their masses, in kilograms, are

320 150 410 280 90 230 360 175 260 125

Each van can carry at most 900 kg.

a. Calculate a lower bound for the number of vans needed.

b. Use the first-fit decreasing algorithm to allocate the parcels to vans.

c. Explain why your allocation is optimal.

d. A late parcel arrives. Using your allocation, find the greatest mass it could have so that no extra van is needed.

(8 marks)

a)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 320 + 150 + 410 + 280 + 90 + 230 + 360 + 175 + 260 + 125 = 2400$$

$$\frac{2400}{900} = 2.667$$

$$\text{Lower bound} = 3$$

3 vans

b) Descending order

410 360 320 280 260 230 175 150 125 90

410: Bin 1

360: Bin 1

320: too big for Bin 1 (space 130), so Bin 2

280: too big for Bin 1 (space 130), so Bin 2

260: too big for Bin 1 (space 130), so Bin 2

230: too big for Bin 1 (space 130), Bin 2 (space 40), so Bin 3

175: too big for Bin 1 (space 130), Bin 2 (space 40), so Bin 3

150: too big for Bin 1 (space 130), Bin 2 (space 40), so Bin 3

125: Bin 1

90: too big for Bin 1 (space 5), Bin 2 (space 40), so Bin 3

Bin 1: 410 360 125 (total 895, space 5)

Bin 2: 320 280 260 (total 860, space 40)

Bin 3: 230 175 150 90 (total 645, space 255)

c)

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3 vans = lower bound 3

Optimal

d) Spare capacity in each van

Van 1: $900 - 895 = 5$

Van 2: $900 - 860 = 40$

Van 3: $900 - 645 = 255$

255 kg

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11 The numbers below are to be packed into bins of size 40

23 8 31 17 26 12 35 5 19 14

a. Use a quick sort to put the numbers into descending order. You should show the result of each pass and identify your pivots clearly.

b. Use the first-fit decreasing algorithm to pack the numbers into bins of size 40

c. Calculate a lower bound for the number of bins and determine whether your packing is optimal. (9 marks)

a) Descending: bigger numbers left of the pivot

Pivot = middle item (right of the middle for an even number)

Pass 1: pivot 12

23 31 17 26 35 19 14 [12] 8 5

Pass 2: pivots 26, 5

31 35 [26] 23 17 19 14 [12] 8 [5]

Pass 3: pivots 35, 19

[35] 31 [26] 23 [19] 17 14 [12] 8 [5]

Pass 4: pivot 14

[35] 31 [26] 23 [19] 17 [14] [12] 8 [5]

35 31 26 23 19 17 14 12 8 5

b) First-fit on the sorted list

35: Bin 1

31: too big for Bin 1 (space 5), so Bin 2

26: too big for Bin 1 (space 5), Bin 2 (space 9), so Bin 3

23: too big for Bin 1 (space 5), Bin 2 (space 9), Bin 3 (space 14), so Bin 4

19: too big for Bin 1 (space 5), Bin 2 (space 9), Bin 3 (space 14), Bin 4 (space 17), so Bin 5

17: too big for Bin 1 (space 5), Bin 2 (space 9), Bin 3 (space 14), so Bin 4

14: too big for Bin 1 (space 5), Bin 2 (space 9), so Bin 3

12: too big for Bin 1 (space 5), Bin 2 (space 9), Bin 3 (space 0), Bin 4 (space 0), so Bin 5

8: too big for Bin 1 (space 5), so Bin 2

5: Bin 1

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Bin 1: 35 5 (total 40, space 0)

Bin 2: 31 8 (total 39, space 1)

Bin 3: 26 14 (total 40, space 0)

Bin 4: 23 17 (total 40, space 0)

Bin 5: 19 12 (total 31, space 9)

c)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 23 + 8 + 31 + 17 + 26 + 12 + 35 + 5 + 19 + 14 = 190$$

$$\frac{190}{40} = 4.750$$

$$\text{Lower bound} = 5$$

5 bins used, lower bound 5

Optimal

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12 A local radio station has ten adverts to broadcast. Their lengths, in seconds, are

75 35 55 75 15 50 85 50 60 40

The adverts are to be placed in advert breaks, each of which lasts at most 3 minutes.

a. Use the first-fit decreasing algorithm to allocate the adverts to breaks.

b. Calculate a lower bound for the number of breaks.

c. Use full-bin packing to show that the adverts can be broadcast in fewer breaks than your answer to part (a).

d. Explain what this shows about the first-fit decreasing algorithm.

(8 marks)

a) Bin size = 3 × 60 = 180 seconds

Descending order

85 75 75 60 55 50 50 40 35 15

85: Bin 1

75: Bin 1

75: too big for Bin 1 (space 20), so Bin 2

60: too big for Bin 1 (space 20), so Bin 2

55: too big for Bin 1 (space 20), Bin 2 (space 45), so Bin 3

50: too big for Bin 1 (space 20), Bin 2 (space 45), so Bin 3

50: too big for Bin 1 (space 20), Bin 2 (space 45), so Bin 3

40: too big for Bin 1 (space 20), so Bin 2

35: too big for Bin 1 (space 20), Bin 2 (space 5), Bin 3 (space 25), so Bin 4

15: Bin 1

Bin 1: 85 75 15 (total 175, space 5)

Bin 2: 75 60 40 (total 175, space 5)

Bin 3: 55 50 50 (total 155, space 25)

Bin 4: 35 (total 35, space 145)

4 breaks

b)

Lower bound = total ÷ bin size, rounded up

$$\text{total} = 75 + 35 + 55 + 75 + 15 + 50 + 85 + 50 + 60 + 40$$

$$= 540$$

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$$\frac{540}{180} = 3.000$$

Lower bound = 3

3 breaks

c) Look for combinations making exactly 180

$85 + 60 + 35 = 180$

$75 + 55 + 50 = 180$

$75 + 50 + 40 + 15 = 180$

All ten adverts used

3 breaks, each exactly 3 minutes

d) First-fit decreasing used 4 but 3 is possible

It is a heuristic: not always optimal

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