Dijkstra's algorithm

Worked solutions

Further Maths topic practice

This topic online

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1 The network below shows the weights of the arcs joining six vertices, A to F.

Use Dijkstra's algorithm to find the shortest path from A to F. State the path and its length.

You should show, at each vertex, the order of labelling, the final value and the working values.

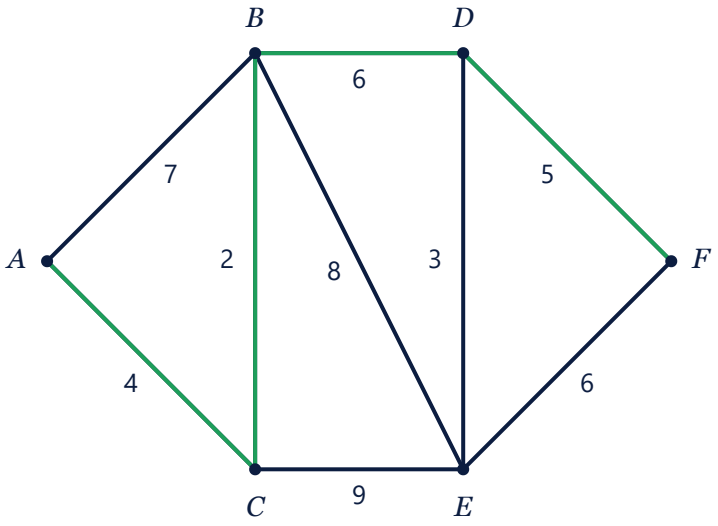


Diagram NOT accurately drawn

(5 marks)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 7 = 7, C gets 0 + 4 = 4

Label C: order 2, final value 4

From C: B gets 4 + 2 = 6, E gets 4 + 9 = 13

Label B: order 3, final value 6

From B: D gets 6 + 6 = 12

Label D: order 4, final value 12

From D: F gets 12 + 5 = 17

Label E: order 5, final value 13

Label F: order 6, final value 17

Working values written in each box

A: 0

B: 7, 6

C: 4

D: 12

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E: 13

F: 17

Trace back from F: an arc is on the route when the difference in final values equals its weight

F 17 - OF 5 = 12 at O

O 12 - BO 6 = 6 at B

B 6 - CB 2 = 4 at C

C 4 - AC 4 = 0 at A

ACBOF, length 17

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2 The network below shows footpaths in a country park in the Peak District. The number on each arc is the time, in minutes, to walk along that path.

Priya walks from the car park, S, to the tea room, T.

a. Use Dijkstra's algorithm to find the quickest route from S to T.

b. State the time taken.

You should show, at each vertex, the order of labelling, the final value and the working values.

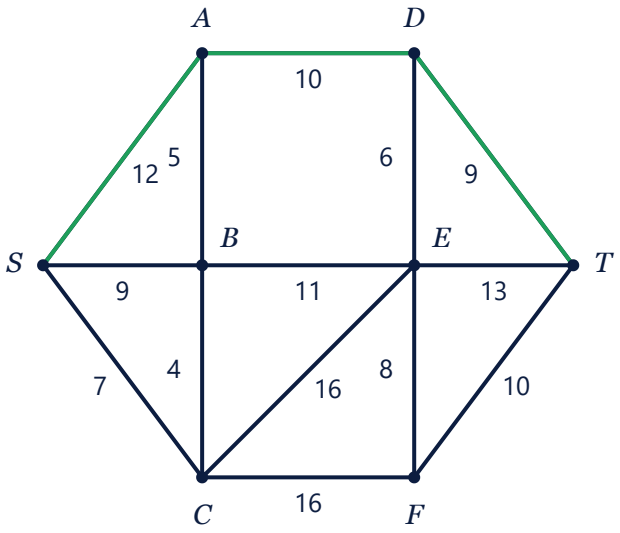


Diagram NOT accurately drawn

(6 marks)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label S: order 1, final value 0

From S: A gets 0 + 12 = 12, B gets 0 + 9 = 9, C gets 0 + 7 = 7

Label C: order 2, final value 7

From C: F gets 7 + 16 = 23, E gets 7 + 16 = 23

Label B: order 3, final value 9

From B: E gets 9 + 11 = 20

Label A: order 4, final value 12

From A: D gets 12 + 10 = 22

Label E: order 5, final value 20

From E: T gets 20 + 13 = 33

Label D: order 6, final value 22

From D: T gets 22 + 9 = 31

Label F: order 7, final value 23

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Label T: order 8, final value 31

Working values written in each box

S: 0

A: 12

B: 9

C: 7

D: 22

E: 23, 20

F: 23

T: 33, 31

Trace back from T: an arc is on the route when the difference in final values equals its weight

T 31 - DT 9 = 22 at D

D 22 - AD 10 = 12 at A

A 12 - SA 12 = 0 at S

SADT, 31 minutes

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3 The network below shows streets in a town centre in Norfolk. The number on each arc is the driving time, in minutes. Arcs with an arrow are one-way streets and may only be used in the direction shown.

a. Use Dijkstra's algorithm to find the quickest route from A to G and state its time.

b. Explain why the quickest route from G back to A is not necessarily the route found in part (a) in reverse.

You should show, at each vertex, the order of labelling, the final value and the working values.

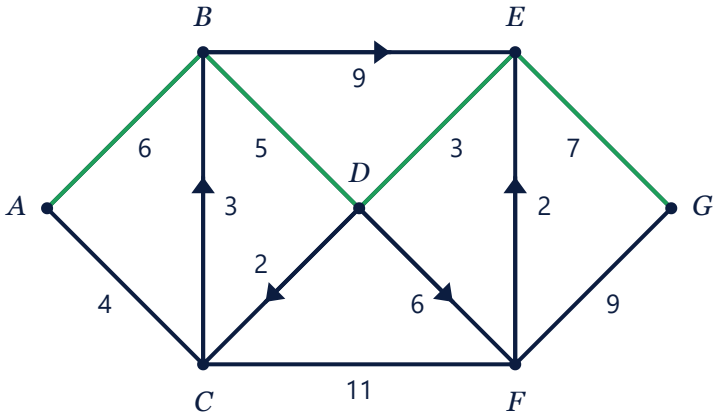


Diagram NOT accurately drawn

(7 marks)

a)

One-way arcs can only be used in the direction of the arrow

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 6 = 6, C gets 0 + 4 = 4

Label C: order 2, final value 4

From C: F gets 4 + 11 = 15

Label B: order 3, final value 6

From B: D gets 6 + 5 = 11, E gets 6 + 9 = 15

Label D: order 4, final value 11

From D: E gets 11 + 3 = 14

Label E: order 5, final value 14

From E: G gets 14 + 7 = 21

Label F: order 6, final value 15

Label G: order 7, final value 21

Working values written in each box

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A: 0

B: 6

C: 4

D: 11

E: 15, 14

F: 15

G: 21

Trace back from G: an arc is on the route when the difference in final values equals its weight

G 21 - EG 7 = 14 at E

E 14 - DE 3 = 11 at D

D 11 - BD 5 = 6 at B

B 6 - AB 6 = 0 at A

ABDEG, 21 minutes

b)

Some streets are one-way (for example BE and DF)

so the arcs that can be used from G to A are different

From G: shortest route back is GEDCA, 16 minutes

One-way streets: not the same arcs in reverse

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4 The network below shows the lengths, in km, of roads between eight towns, A to H, in Yorkshire.

a. Use Dijkstra's algorithm to find the shortest route from A to H. State the route and its length.

Dijkstra's algorithm has order n^2, where n is the number of vertices. A computer takes 0.6 seconds to apply the algorithm to a network with 40 vertices.

b. Estimate the time the computer would take for a network with 250 vertices.

You should show, at each vertex, the order of labelling, the final value and the working values.

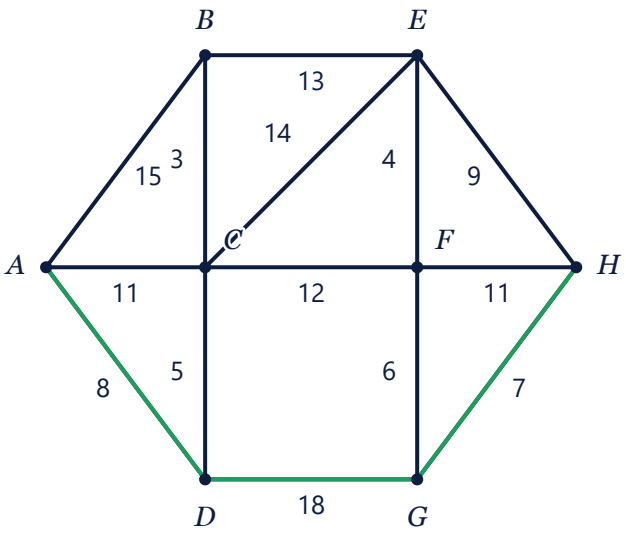


Diagram NOT accurately drawn

(8 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 15 = 15, C gets 0 + 11 = 11, D gets 0 + 8 = 8

Label D: order 2, final value 8

From D: G gets 8 + 18 = 26

Label C: order 3, final value 11

From C: B gets 11 + 3 = 14, F gets 11 + 12 = 23, E gets 11 + 14 = 25

Label B: order 4, final value 14

Label F: order 5, final value 23

From F: H gets 23 + 11 = 34

Label E: order 6, final value 25

Label G: order 7, final value 26

From G: H gets 26 + 7 = 33

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Label H: order 8, final value 33

Working values written in each box

A: 0

B: 15, 14

C: 11

D: 8

E: 25

F: 23

G: 26

H: 34, 33

Trace back from H: an arc is on the route when the difference in final values equals its weight

H 33 - GH 7 = 26 at G

G 26 - DG 18 = 8 at D

D 8 - AD 8 = 0 at A

ADGH, 33 km

b)

Order n^2: time $\propto n^2$

$$\text{Scale factor} = \frac{250}{40} = 6.25$$

$$\text{Time} = 0.6 \times 6.25^2$$

$$\text{Time} = 0.6 \times 39.0625$$

$$\text{Time} = 23.4375$$

23.4 seconds

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5 The network below shows the lengths, in miles, of roads between eight villages, A to H, in Norfolk. Sam drives a mobile library van from A to H and must stop at village C on the way.

a. Apply Dijkstra's algorithm, starting at C, to find the shortest distance from C to every other village.

b. Hence find the shortest route from A to H that passes through C, and its length.

c. The shortest route from A to H with no stop at C is 62 miles. Find how much further Sam drives because of the stop at C.

You should show, at each vertex, the order of labelling, the final value and the working values.

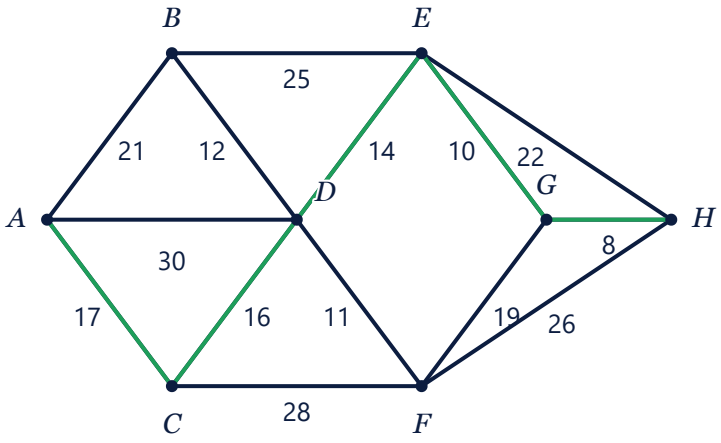


Diagram NOT accurately drawn

(8 marks)

a)

The roads are two-way, so C to A is the same as A to C

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label C: order 1, final value 0

From C: A gets 0 + 17 = 17, D gets 0 + 16 = 16, F gets 0 + 28 = 28

Label D: order 2, final value 16

From D: B gets 16 + 12 = 28, E gets 16 + 14 = 30, F gets 16 + 11 = 27

Label A: order 3, final value 17

Label F: order 4, final value 27

From F: G gets 27 + 19 = 46, H gets 27 + 26 = 53

Label B: order 5, final value 28

Label E: order 6, final value 30

From E: G gets 30 + 10 = 40, H gets 30 + 22 = 52

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Label G: order 7, final value 40

From G: H gets 40 + 8 = 48

Label H: order 8, final value 48

Working values written in each box

A: 17

B: 28

C: 0

D: 16

E: 30

F: 28, 27

G: 46, 40

H: 53, 52, 48

D 16, A 17, F 27, B 28, E 30, G 40, H 48

b)

Trace back from A and from H to C

A 17: CA direct (17)

H 48 - GH 8 = 40 at G

G 40 - EG 10 = 30 at E

E 30 - DE 14 = 16 at D

D 16 - CD 16 = 0 at C

Length = 17 + 48 = 65

ACDEGH; 65 miles

c)

Extra = 65 - 62 = 3

3 miles

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6 The table shows the lengths, in km, of the direct cycle routes between seven towns, A to G, in Kent. A dash means there is no direct route.

	A	B	C	D	E	F	G
A	–	24	18	35	–	–	–
B	24	–	9	–	20	–	–
C	18	9	–	14	27	–	–
D	35	–	14	–	–	16	40
E	–	20	27	–	–	8	25
F	–	–	–	16	8	–	19
G	–	–	–	40	25	19	–

Use Dijkstra's algorithm to find the shortest route from A to G and state its length. You may find it helpful to draw the network first.

You should show, at each vertex, the order of labelling, the final value and the working values. (6 marks)

Each number in the table is a direct route; a dash means none

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 24 = 24, C gets 0 + 18 = 18, D gets 0 + 35 = 35

Label C: order 2, final value 18

From C: D gets 18 + 14 = 32, E gets 18 + 27 = 45

Label B: order 3, final value 24

From B: E gets 24 + 20 = 44

Label D: order 4, final value 32

From D: F gets 32 + 16 = 48, G gets 32 + 40 = 72

Label E: order 5, final value 44

From E: G gets 44 + 25 = 69

Label F: order 6, final value 48

From F: G gets 48 + 19 = 67

Label G: order 7, final value 67

Working values written in each box

A: 0

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B: 24

C: 18

O: 35, 32

E: 45, 44

F: 48

G: 72, 69, 67

Trace back from G: an arc is on the route when the difference in final values equals its weight

G 67 - FG 19 = 48 at F

F 48 - OF 16 = 32 at O

O 32 - CO 14 = 18 at C

C 18 - AC 18 = 0 at A

ACOFG, 67 km

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7 The network below shows the times, in minutes, for a bus to travel between stops S, A, B, C, D, E, F and T in a city in Wales.

a. Use Dijkstra's algorithm to find the quickest route from S to T and state its time.

One week, stop B is closed for building work and the bus cannot pass through it.

b. Use Dijkstra's algorithm again to find the quickest route from S to T that avoids B.

c. State the increase in journey time.

You should show, at each vertex, the order of labelling, the final value and the working values.

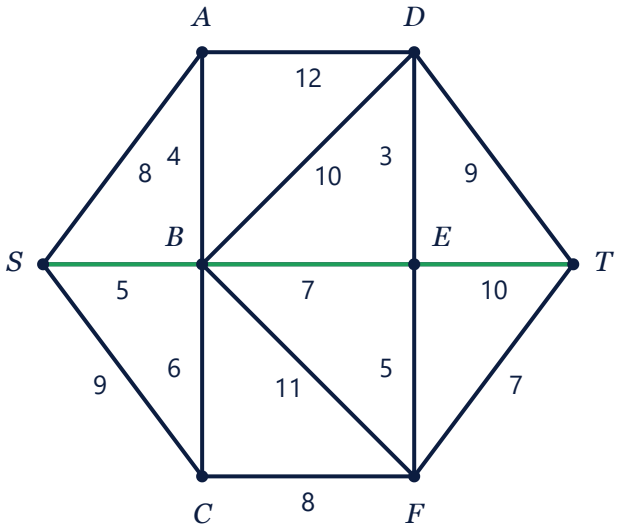


Diagram NOT accurately drawn

(9 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label S: order 1, final value 0

From S: A gets 0 + 8 = 8, B gets 0 + 5 = 5, C gets 0 + 9 = 9

Label B: order 2, final value 5

From B: D gets 5 + 10 = 15, E gets 5 + 7 = 12, F gets 5 + 11 = 16

Label A: order 3, final value 8

Label C: order 4, final value 9

Label E: order 5, final value 12

From E: T gets 12 + 10 = 22

Label D: order 6, final value 15

Label F: order 7, final value 16

Label T: order 8, final value 22

Working values written in each box

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S: 0

A: 8

B: 5

C: 9

D: 15

E: 12

F: 16

T: 22

Trace back from T: an arc is on the route when the difference in final values equals its weight

T 22 - ET 10 = 12 at E

E 12 - BE 7 = 5 at B

B 5 - SB 5 = 0 at S

SBET, 22 minutes

b)

Remove B and all arcs at B: SB, AB, BC, BD, BE, BF

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label S: order 1, final value 0

From S: A gets 0 + 8 = 8, C gets 0 + 9 = 9

Label A: order 2, final value 8

From A: D gets 8 + 12 = 20

Label C: order 3, final value 9

From C: F gets 9 + 8 = 17

Label F: order 4, final value 17

From F: E gets 17 + 5 = 22, T gets 17 + 7 = 24

Label D: order 5, final value 20

Label E: order 6, final value 22

Label T: order 7, final value 24

Working values written in each box

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S: 0

A: 8

C: 9

D: 20

F: 17

E: 22

T: 24

Trace back from T: an arc is on the route when the difference in final values equals its weight

T 24 - FT 7 = 17 at F

F 17 - CF 8 = 9 at C

C 9 - SC 9 = 0 at S

SCFT, 24 minutes

c)

Increase = 24 - 22 = 2

2 minutes

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8 The network below shows the times, in seconds, taken by a warehouse robot to travel between points A to H.

a. Use Dijkstra's algorithm to find the quickest route from A to H and state its time.

A stack of pallets slows the robot down on DF, so the time for DF increases from 16 seconds to 34 seconds.

b. Find the new quickest route from A to H and its time.

c. State how much longer the new quickest route takes.

You should show, at each vertex, the order of labelling, the final value and the working values.

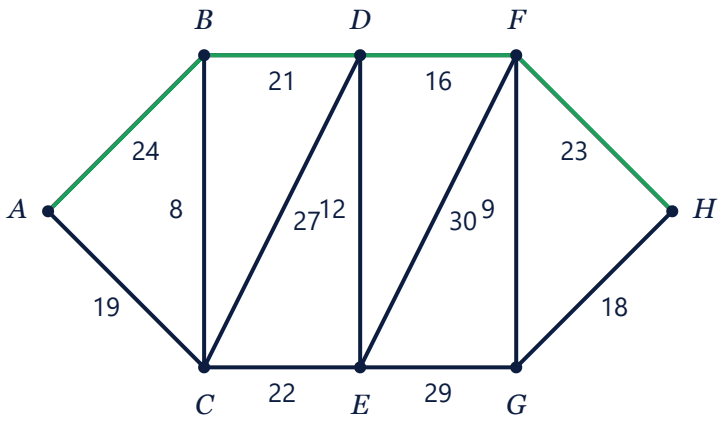


Diagram NOT accurately drawn

(9 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 24 = 24, C gets 0 + 19 = 19

Label C: order 2, final value 19

From C: D gets 19 + 27 = 46, E gets 19 + 22 = 41

Label B: order 3, final value 24

From B: D gets 24 + 21 = 45

Label E: order 4, final value 41

From E: G gets 41 + 29 = 70, F gets 41 + 30 = 71

Label D: order 5, final value 45

From D: F gets 45 + 16 = 61

Label F: order 6, final value 61

From F: H gets 61 + 23 = 84

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Label G: order 7, final value 70

Label H: order 8, final value 84

Working values written in each box

A: 0

B: 24

C: 19

D: 46, 45

E: 41

F: 71, 61

G: 70

H: 84

Trace back from H: an arc is on the route when the difference in final values equals its weight

H 84 - FH 23 = 61 at F

F 61 - DF 16 = 45 at D

D 45 - BD 21 = 24 at B

B 24 - AB 24 = 0 at A

ABDFH, 84 seconds

b)

Use DF = 34 and apply the algorithm again

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 24 = 24, C gets 0 + 19 = 19

Label C: order 2, final value 19

From C: D gets 19 + 27 = 46, E gets 19 + 22 = 41

Label B: order 3, final value 24

From B: D gets 24 + 21 = 45

Label E: order 4, final value 41

From E: G gets 41 + 29 = 70, F gets 41 + 30 = 71

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Label D: order 5, final value 45

Label G: order 6, final value 70

From G: H gets 70 + 18 = 88

Label F: order 7, final value 71

Label H: order 8, final value 88

Working values written in each box

A: 0

B: 24

C: 19

D: 46, 45

E: 41

F: 71

G: 70

H: 88

Trace back from H: an arc is on the route when the difference in final values equals its weight

H 88 - GH 18 = 70 at G

G 70 - EG 29 = 41 at E

E 41 - CE 22 = 19 at C

C 19 - AC 19 = 0 at A

ACEGH, 88 seconds

c)

Difference = 88 - 84 = 4

4 seconds

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9 The network below shows the distances, in hundreds of metres, between seven points, A to G, on a hospital site in Manchester.

a. Use Dijkstra's algorithm to find the shortest distance from A to G.

b. There is more than one shortest route from A to G. State every shortest route.

You should show, at each vertex, the order of labelling, the final value and the working values.

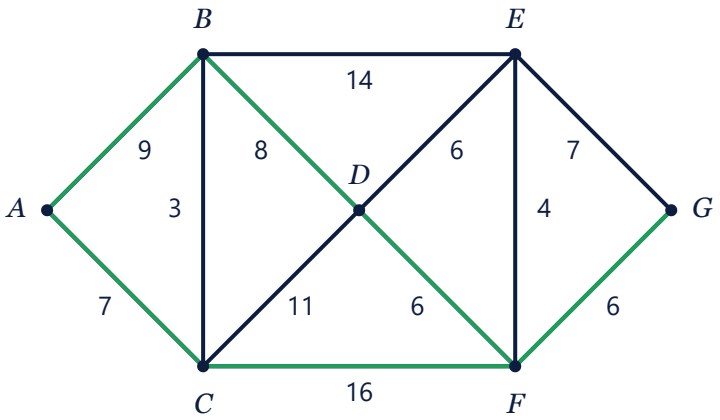


Diagram NOT accurately drawn

(7 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 9 = 9, C gets 0 + 7 = 7

Label C: order 2, final value 7

From C: D gets 7 + 11 = 18, F gets 7 + 16 = 23

Label B: order 3, final value 9

From B: D gets 9 + 8 = 17, E gets 9 + 14 = 23

Label D: order 4, final value 17

Label E: order 5, final value 23

From E: G gets 23 + 7 = 30

Label F: order 6, final value 23

From F: G gets 23 + 6 = 29

Label G: order 7, final value 29

Working values written in each box

A: 0

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B: 9

C: 7

O: 18, 17

E: 23

F: 23

G: 30, 29

Trace back from G: check every arc into each vertex

G 29: FG gives 23 + 6 = 29, so G comes from F

EG gives 23 + 7 = 30, too big

F 23: OF gives 17 + 6 = 23 and CF gives 7 + 16 = 23: both work

O 17: BO gives 9 + 8 = 17, so B

B 9: AB gives 0 + 9 = 9, so A

C 7: AC gives 0 + 7 = 7, so A

29

b)

ABOFG and ACFG

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10 The network below shows the lengths, in km, of tracks in a forest in Hampshire. The length of track CF is x km.

a. Given that $x = 20$, use Dijkstra's algorithm to find the shortest route from A to G and state its length.

b. Find the range of values of x for which CF is on the shortest route from A to G.

You should show, at each vertex, the order of labelling, the final value and the working values.

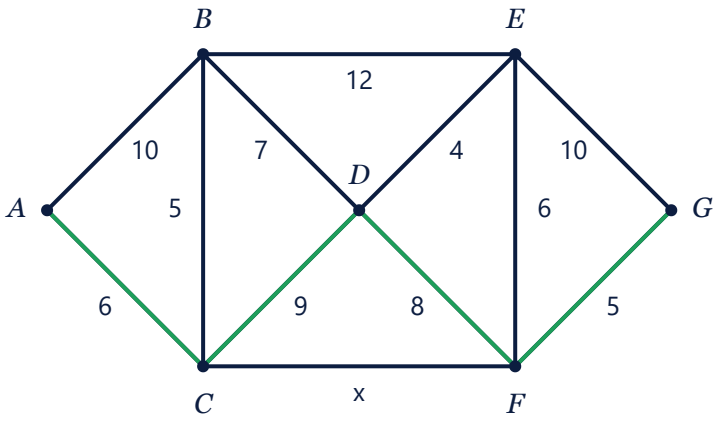


Diagram NOT accurately drawn

(8 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label A: order 1, final value 0

From A: B gets 0 + 10 = 10, C gets 0 + 6 = 6

Label C: order 2, final value 6

From C: D gets 6 + 9 = 15, F gets 6 + 20 = 26

Label B: order 3, final value 10

From B: E gets 10 + 12 = 22

Label D: order 4, final value 15

From D: E gets 15 + 4 = 19, F gets 15 + 8 = 23

Label E: order 5, final value 19

From E: G gets 19 + 10 = 29

Label F: order 6, final value 23

From F: G gets 23 + 5 = 28

Label G: order 7, final value 28

Working values written in each box

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A: 0

B: 10

C: 6

D: 15

E: 22, 19

F: 26, 23

G: 29, 28

Trace back from G: an arc is on the route when the difference in final values equals its weight

G 28 - FG 5 = 23 at F

F 23 - DF 8 = 15 at D

D 15 - CD 9 = 6 at C

C 6 - AC 6 = 0 at A

ACDFG, 28 km

b)

Best route that uses CF

Shortest A to C is AC = 6

Shortest F to G is FG = 5

Length via CF = $6 + x + 5 = 11 + x$

This must be shorter than the best route without CF

$11 + x < 28$

$x < 17$

x is a length so $x > 0$

$0 < x < 17$

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11 An ambulance is based at a hospital, H, in rural Northumberland. The network below shows the driving times, in minutes, between H and eight villages, A to G and J.

a. Use Dijkstra's algorithm, starting at H, to find the shortest driving time from H to every village.

b. State the quickest route from H to J and its time.

c. The ambulance service aims to reach every village within 30 minutes. State, with a reason, whether this aim is met.

d. Explain why one application of Dijkstra's algorithm is enough to answer parts (b) and (c).

You should show, at each vertex, the order of labelling, the final value and the working values.

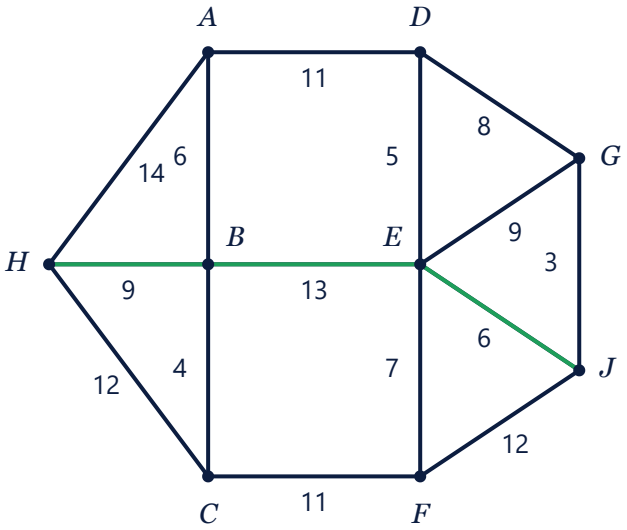


Diagram NOT accurately drawn

(9 marks)

a)

Carry on until every vertex is labelled

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label H: order 1, final value 0

From H: A gets 0 + 14 = 14, B gets 0 + 9 = 9, C gets 0 + 12 = 12

Label B: order 2, final value 9

From B: E gets 9 + 13 = 22

Label C: order 3, final value 12

From C: F gets 12 + 11 = 23

Label A: order 4, final value 14

From A: D gets 14 + 11 = 25

Label E: order 5, final value 22

From E: G gets 22 + 9 = 31, J gets 22 + 6 = 28

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Label F: order 6, final value 23

Label D: order 7, final value 25

Label J: order 8, final value 28

Label G: order 9, final value 31

Working values written in each box

H: 0

A: 14

B: 9

C: 12

D: 25

E: 22

F: 23

G: 31

J: 28

B 9, C 12, A 14, E 22, F 23, D 25, J 28, G 31

b)

Trace back from J

J 28 - EJ 6 = 22 at E

E 22 - BE 13 = 9 at B

B 9 - HB 9 = 0 at H

HBEJ, 28 minutes

c)

Largest final value: G = 31

31 > 30

Not met: G takes 31 minutes

d)

Every final value is the shortest time from H to that vertex

One run from H gives all the times

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12 The network below shows the lengths, in miles, of roads between ten places in Cumbria. A delivery driver starts at the depot, D.

a. Use Dijkstra's algorithm to find the shortest route from D to K and state its length.

On Friday the driver must collect a parcel from E on the way from D to K.

b. Use your answer to part (a) to find the shortest route from D to E.

c. Hence find the shortest route from D to K that passes through E, and its length.

d. Find how many extra miles the driver travels on Friday.

You should show, at each vertex, the order of labelling, the final value and the working values.

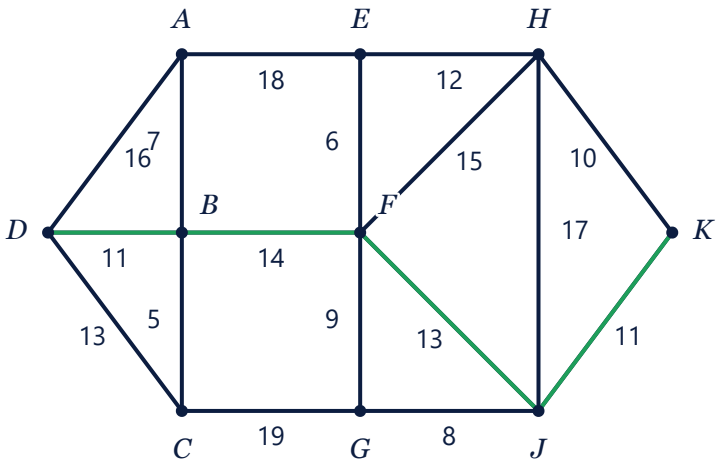


Diagram NOT accurately drawn

(10 marks)

a)

Each box: order of labelling, final value, working values

Next label = smallest working value not yet final

Label D: order 1, final value 0

From D: A gets 0 + 16 = 16, B gets 0 + 11 = 11, C gets 0 + 13 = 13

Label B: order 2, final value 11

From B: F gets 11 + 14 = 25

Label C: order 3, final value 13

From C: G gets 13 + 19 = 32

Label A: order 4, final value 16

From A: E gets 16 + 18 = 34

Label F: order 5, final value 25

From F: E gets 25 + 6 = 31, H gets 25 + 15 = 40, J gets 25 + 13 = 38

Label E: order 6, final value 31

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Label G: order 7, final value 32

Label J: order 8, final value 38

From J: K gets 38 + 11 = 49

Label H: order 9, final value 40

Label K: order 10, final value 49

Working values written in each box

O: 0

A: 16

B: 11

C: 13

E: 34, 31

F: 25

G: 32

H: 40

J: 38

K: 49

Trace back from K: an arc is on the route when the difference in final values equals its weight

K 49 - JK 11 = 38 at J

J 38 - FJ 13 = 25 at F

F 25 - BF 14 = 11 at B

B 11 - OB 11 = 0 at O

OBFJK, 49 miles

b)

E was labelled in part (a): trace back from E

E 31 - FE 6 = 25 at F

F 25 - BF 14 = 11 at B

B 11 - OB 11 = 0 at O

OBFE, 31 miles

c)

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Shortest E to K by inspection

EHK = 12 + 10 = 22

EFJK = 6 + 13 + 11 = 30

EFHK = 6 + 15 + 10 = 31

so E to K is EHK, 22

Length = 31 + 22 = 53

OBFEHK, 53 miles

d)

Extra = 53 - 49 = 4

4 miles

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