Graphs and networks

Worked solutions

Further Maths topic practice

This topic online

1 The diagram shows a graph G.

a. Write down the degree of each vertex of G.

b. Show that your answers agree with the handshaking lemma, and explain why the lemma is true.

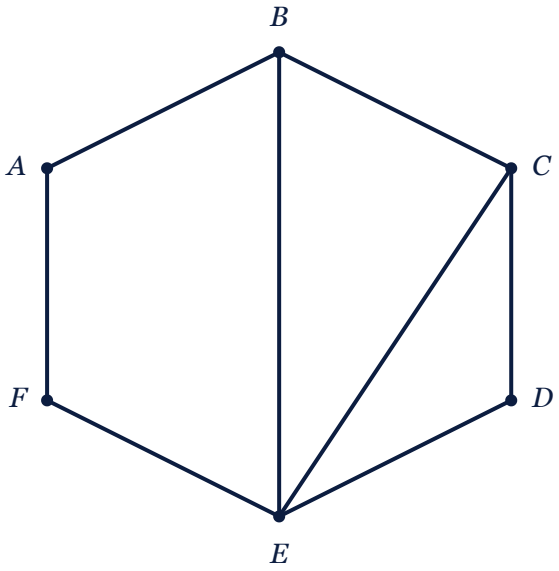


Diagram NOT accurately drawn

(3 marks)

a) Count the edges at each vertex

deg(A) = 2

deg(B) = 3

deg(C) = 3

deg(D) = 2

deg(E) = 4

deg(F) = 2

b) Handshaking lemma: sum of degrees = 2 × number of edges

sum = 2 + 3 + 3 + 2 + 4 + 2 = 16

2 × 8 = 16

Each edge has two ends, so it adds 2 to the total

16 = 2 × 8

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2 For the graph shown, write down

a. a path from P to S that passes through exactly two other vertices,

b. a cycle that passes through exactly four vertices,

c. a Hamiltonian cycle.

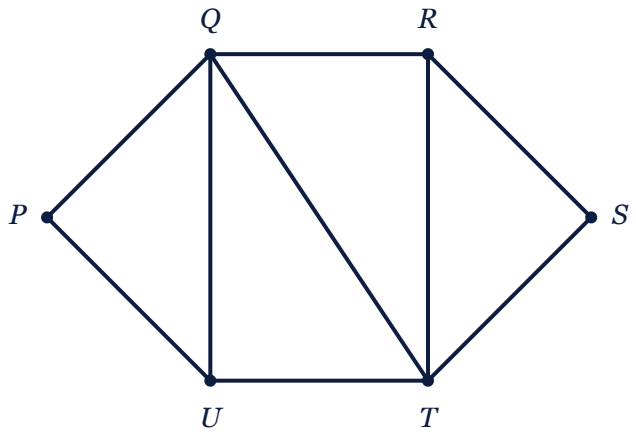


Diagram NOT accurately drawn

(3 marks)

a) Path: no vertex repeated

P - U - T - S (or P - Q - R - S)

P U T S

b) Cycle: closed path, start = end

Q - R - T - U - Q uses edges QR, RT, TU, UQ

Q R T U Q

c) Hamiltonian cycle: every vertex once, back to the start

P - Q - R - S - T - U - P

P Q R S T U P

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3 K_n denotes the complete graph on n vertices.

a. Show that K_n has $\frac{1}{2}n(n-1)$ edges.

b. Find the number of edges in K_7

c. A simple graph has 7 vertices and 15 edges. Find how many edges must be added to make it K_7 (4 marks)

a) Every vertex joins all the others

degree of each vertex = $n - 1$

sum of degrees = $n(n-1)$

Handshaking lemma: sum = 2 × edges

edges = $\frac{1}{2}n(n-1)$

b) $n = 7$

edges = $\frac{1}{2} \times 7 \times 6$

edges = 21

21

c)

$21 - 15 = 6$

6 edges

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4 The diagram shows a graph with five vertices. There are two edges joining A and B.

a. Write down the adjacency matrix for the graph.

b. Explain how the degree of each vertex can be found from the matrix, and state the degree of B.

c. Explain why the graph is not a simple graph.

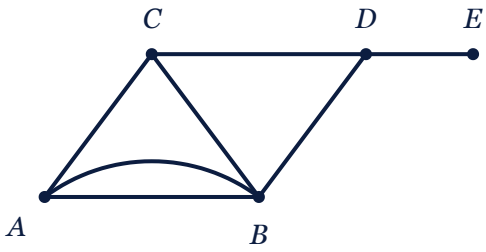


Diagram NOT accurately drawn

(4 marks)

a) Entry = number of edges joining the two vertices

columns: A B C D E

A: 0 2 1 0 0

B: 2 0 1 1 0

C: 1 1 0 1 0

D: 0 1 1 0 1

E: 0 0 0 1 0

Check: the matrix is symmetric

b) Degree = row total

$\deg(B) = 2 + 0 + 1 + 1 + 0 = 4$

$\deg(B) = 4$

c) Simple: no loops, no multiple edges

A and B are joined by two edges

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5 The table shows the lengths, in km, of the roads directly joining five villages A, B, C, D and E. A dash means there is no direct road.

	A	B	C	D	E
A	–	12	9	–	15
B	12	–	7	10	–
C	9	7	–	8	11
D	–	10	8	–	6
E	15	–	11	6	–

a. Using the vertices in the diagram, draw the network, labelling each edge with its length.

b. Find the total length of the roads.

c. Write down the degree of each vertex.

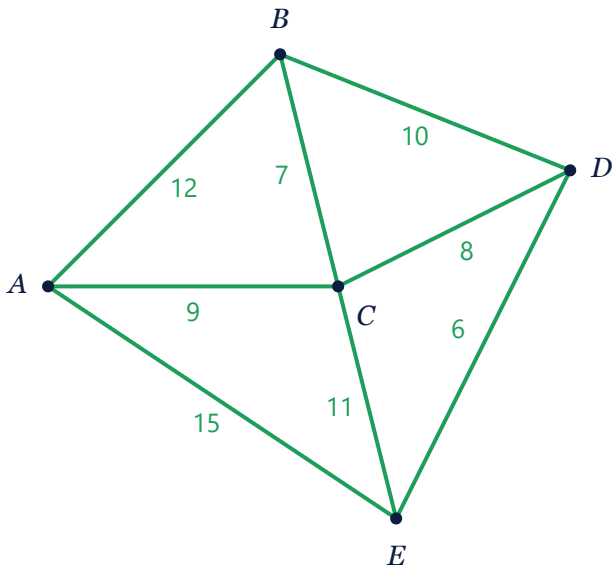


Diagram NOT accurately drawn

(4 marks)

a) Read the table above the diagonal: one edge per number

AB = 12

AC = 9

AE = 15

BC = 7

BD = 10

CD = 8

CE = 11

DE = 6

Network drawn (shown in green)

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b) Add the edge lengths

$$\text{total} = 12 + 9 + 15 + 7 + 10 + 8 + 11 + 6$$

$$\text{total} = 78 \text{ km}$$

78 km

c) Count the numbers in each row

$\deg(A) = 3$

$\deg(B) = 3$

$\deg(C) = 4$

$\deg(D) = 3$

$\deg(E) = 3$

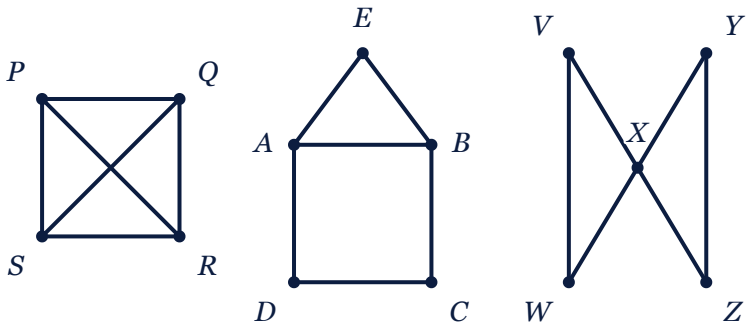
A 3, B 3, C 4, D 3, E 3

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6 The diagram shows three graphs.

a. For each graph, state whether it is Eulerian, semi-Eulerian or neither. You must justify your answers.

b. For the semi-Eulerian graph, state where a route that uses every edge exactly once must start and finish, and write down such a route.



Graph 1 Graph 2 Graph 3

Diagram NOT accurately drawn

(5 marks)

a) Count the odd vertices

Graph 1: P 3, Q 3, R 3, S 3

4 odd vertices

Graph 1: neither

Graph 2: A 3, B 3, C 2, D 2, E 2

exactly 2 odd vertices (A and B)

Graph 2: semi-Eulerian

Graph 3: V 2, W 2, X 4, Y 2, Z 2

all vertices even

Graph 3: Eulerian

b) Start and finish at the two odd vertices

A - E - B - A - D - C - B uses all 6 edges once

Start A, finish B

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7 This question is about the degrees of vertices.

a. Explain why there is no graph whose vertices have degrees 1, 2, 2, 3, 3, 4 and 4

A tree T has seven vertices. The degrees of the vertices are

1 1 1 1 x $2x - 3$ $x - 1$

b. State the number of edges in T.

c. Find the value of x.

d. State, with a reason, whether T is Eulerian, semi-Eulerian or neither.

e. Explain why no tree with more than one vertex can be Eulerian.

(6 marks)

a) Handshaking Lemma: sum of degrees is even

$1 + 2 + 2 + 3 + 3 + 4 + 4 = 19$

19 is odd, so no such graph

b) A tree with n vertices has n - 1 edges

edges = 7 - 1 = 6

6 edges

c) Sum of degrees = 2 x 6 = 12

$1 + 1 + 1 + 1 + x + (2x - 3) + (x - 1) = 12$

$4x = 12$

$x = 3$

$x = 3$

d) Degrees with $x = 3$

1, 1, 1, 1, 3, 3, 2

six odd vertices (more than two)

Neither

e) Every tree has at least two leaves

a leaf has degree 1, which is odd

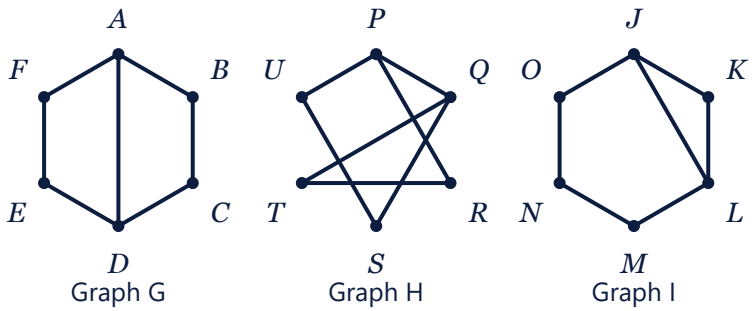
So a tree always has odd vertices

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8 The diagram shows three graphs G, H and I, each with six vertices and seven edges.

a. Show that G and H are isomorphic by writing down which vertex of H corresponds to each vertex of G.

b. Explain why I is not isomorphic to G, even though both graphs have the same degrees.



Graph G Graph H Graph I

Diagram NOT accurately drawn

(5 marks)

a) Match the degree 3 vertices first

G: A and O have degree 3, joined by AO

H: P and Q have degree 3, joined by PQ

Follow the 6-cycle in each graph

G: A - B - C - O - E - F - A

H: P - R - T - Q - S - U - P

A = P, B = R, C = T, O = Q, E = S, F = U

Check: AO matches PQ, and the cycle edges match

G and H are isomorphic

b) Same degrees, so look at cycles

I: J - K - L - J is a cycle of length 3

G: AO splits the hexagon into two 4-cycles, no 3-cycle

I has a triangle, G does not

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9 The diagram shows a graph. Ignore the dashed edge CG in part (a).

a. Explain why the graph without CG has no Hamiltonian cycle.

b. The edge CG is added. Find a Hamiltonian cycle starting at A.

c. State, with a reason, whether the graph with CG added is Eulerian, semi-Eulerian or neither.

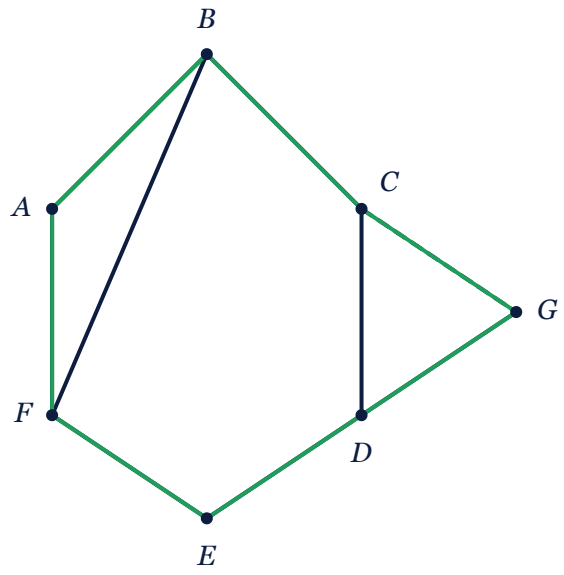


Diagram NOT accurately drawn

(5 marks)

a) A cycle must enter and leave every vertex

G has degree 1 (only edge DG)

No Hamiltonian cycle

b) G must use both CG and DG

A - B - C - G - D - E - F - A

A B C G D E F A

c) Degrees with CG added

deg(A) = 2

deg(B) = 3

deg(C) = 3

deg(D) = 3

deg(E) = 2

deg(F) = 3

deg(G) = 2

odd vertices: B, C, D, F

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Neither: more than two odd vertices

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10 The network shows the footpaths in a country park. The vertices are the car park (C), lake (L), mill (M), tower (T), wood (W) and bridge (B). The weights are the lengths of the paths in km.

a. Find the total length of the footpaths.

b. Explain why the network is semi-Eulerian.

c. A ranger wants to walk along every path exactly once. State where she must start and finish, and write down a suitable route.

d. One path is to be closed so that the ranger can walk along every remaining path exactly once and return to her starting point. State which path, and the length of her walk.

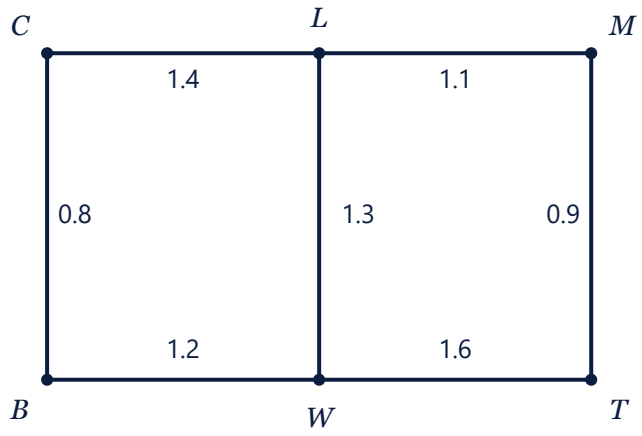


Diagram NOT accurately drawn

(7 marks)

a) Add all the weights

total = 1.4 + 1.1 + 0.9 + 1.6 + 1.2 + 0.8 + 1.3

total = 8.3 km

8.3 km

b) Degrees

deg(C) = 2

deg(L) = 3

deg(M) = 2

deg(B) = 2

deg(W) = 3

deg(T) = 2

Exactly two odd vertices: L and W

Semi-Eulerian

c) Start at one odd vertex, finish at the other

L - C - B - W - T - M - L - W

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Start L, finish W

d) Removing LW makes L and W even

Then every vertex is even: Eulerian

length = 8.3 − 1.3 = 7 km

Close LW: 7 km

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11 The graph shown has six vertices and nine edges.

a. State the number of edges in a spanning tree of this graph, and hence the number of edges that must be removed to leave a spanning tree.

b. Write down the edges of a spanning tree in which vertex A has degree 5

c. Write down the edges of a spanning tree that is a path, starting at B and ending at A.

d. Explain why there is no spanning tree in which every vertex has degree at most 1

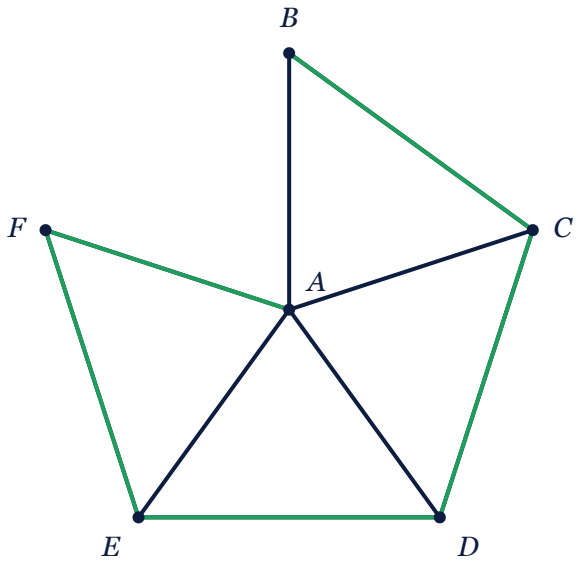


Diagram NOT accurately drawn

(6 marks)

a) Spanning tree on n vertices has n - 1 edges

edges = 6 - 1 = 5

remove 9 - 5 = 4

5 edges, remove 4

b) A is joined to all five other vertices

AB, AC, AD, AE, AF

The star at A

c) Path through every vertex, B to A

B - C - D - E - F - A

BC, CD, DE, EF, FA

d) Handshaking lemma

sum of degrees = 2 × 5 = 10

six vertices of degree ≤ 1 give a sum ≤ 6

6 < 10, so impossible

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12 This question is about the complete graphs K_n, where $n \geq 3$

a. Explain why K_n is Eulerian when n is odd and is neither Eulerian nor semi-Eulerian when n is even.

b. Find the smallest number of edges that must be removed from K_8 to leave an Eulerian graph, and describe which edges could be removed.

c. Find the number of edges in the graph left in part (b).

d. The vertices of K_8 are labelled 1 to 8 and the edges 12, 34, 56 and 78 are removed. Write down a Hamiltonian cycle in the graph that is left.

(8 marks)

a) Each vertex joins the other $n - 1$

degree $= n - 1$

n odd: $n - 1$ even, all vertices even

Eulerian

n even: all n vertices odd, $n \geq 4$

More than two odd vertices: neither

b) K8: every vertex has degree 7 (odd)

Removing an edge makes two vertices even

edges needed $= \frac{8}{2} = 4$

Choose 4 edges with no vertex in common

4 edges, e.g. 12, 34, 56, 78

c) Edges in K8

$\frac{1}{2} \times 8 \times 7 = 28$

$28 - 4 = 24$

24 edges

d) Avoid 12, 34, 56, 78

1 - 3 - 5 - 7 - 2 - 4 - 6 - 8 - 1

1 3 5 7 2 4 6 8 1

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