Linear programming

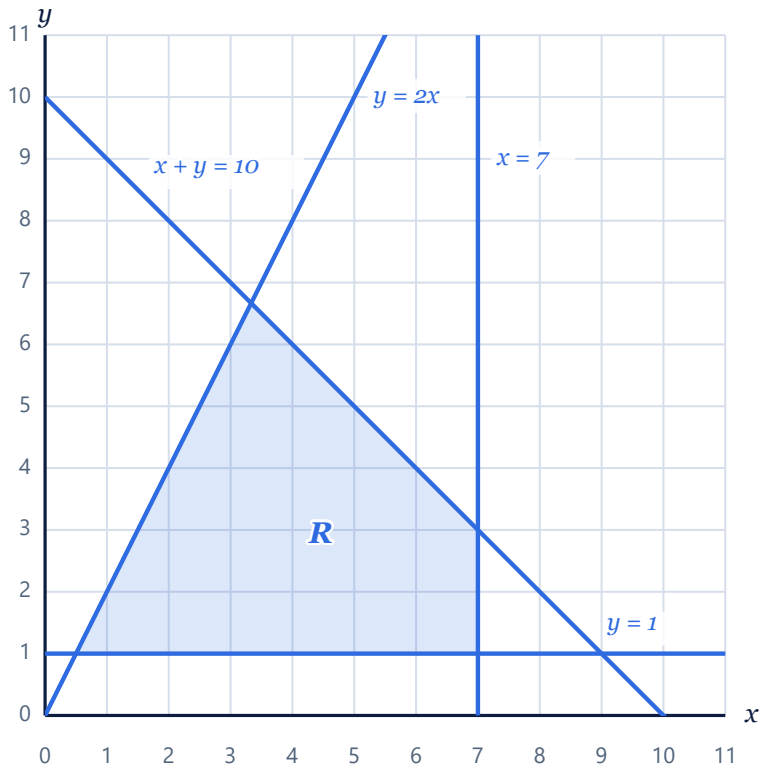
Worked solutions

Further Maths topic practice

This topic online

1 The diagram shows the feasible region, R, of a linear programming problem in x and y.

Write down the inequalities that define R.



(3 marks)

Test a point inside R, for example (4, 3)

$3 \geqslant 1$, so $y \geqslant 1$

$4 + 3 = 7 \leqslant 10$, so $x + y \leqslant 10$

$3 \leqslant 2 \times 4$, so $y \leqslant 2x$

$4 \leqslant 7$, so $x \leqslant 7$

$y \geqslant 1,\ x + y \leqslant 10,\ y \leqslant 2x,\ x \leqslant 7$

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2 A bakery in Cardiff makes x loaves of bread and y cakes each day.

- Each loaf uses 500 g of flour and each cake uses 200 g of flour. There is 20 kg of flour available.
- Each loaf needs 30 minutes of oven time and each cake needs 20 minutes. The oven is available for 15 hours.
- At least 10 cakes must be made.
- The number of loaves must be at least half the number of cakes.

Write down four inequalities, other than $x \geq 0$, to model these constraints. Simplify each inequality so that it has integer coefficients that are as small as possible. (4 marks)

Use the same units on both sides

Flour: $500x + 200y \leq 20000$

Divide by 100

$\boxed{5x + 2y \leq 200}$

Oven: $30x + 20y \leq 15 \times 60$

$30x + 20y \leq 900$

Divide by 10

$\boxed{3x + 2y \leq 90}$

$\boxed{y \geq 10}$

Loaves: $x \geq \frac{1}{2}y$

$\boxed{2x \geq y}$

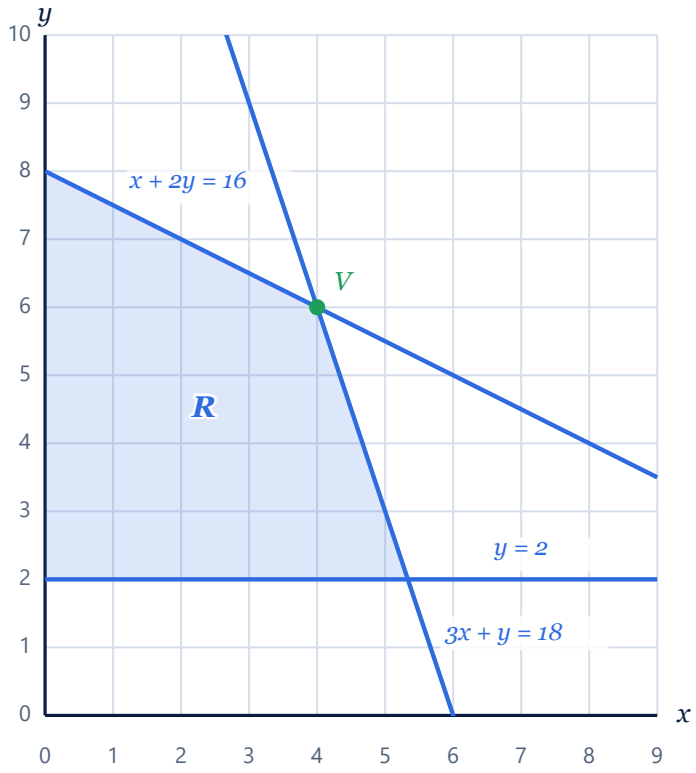
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3 The diagram shows the feasible region, R, for a linear programming problem. R is defined by $x + 2y \leq 16$, $3x + y \leq 18$, $y \geq 2$ and $x \geq 0$

The objective is to maximise $P = 2x + 3y$

a. Find the exact coordinates of each vertex of R.

b. Use the vertex method to find the optimal vertex and the maximum value of P.



(5 marks)

a)

$x = 0$ meets $y = 2$ at $(0, 2)$

Solve $y = 2$ (1) and $3x + y = 18$ (2)

$$y = 2$$

$$3x = 18 - 1 \times 2$$

$$x = \frac{16}{3}$$

Solve $x + 2y = 16$ (1) and $3x + y = 18$ (2)

Eliminate x: (1) $\times 3$, (2) $\times 1$

$$3x + 6y = 48$$

$$3x + y = 18$$

Subtract the second from the first

$$5y = 30$$

$$y = 6$$

Substitute into (1)

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$x = 16 - 2 \times 6$

$x = 4$

Solve $x = 0$ (1) and $x + 2y = 16$ (2)

$x = 0$

$2y = 16 - 1 \times 0$

$y = 8$

$\boxed{(0, 2), (\frac{16}{3}, 2), (4, 6), (0, 8)}$

b)

Evaluate $P = 2x + 3y$ at each vertex

At $(0, 2)$: $P = 2 \times 0 + 3 \times 2 = 6$

At $(\frac{16}{3}, 2)$: $P = 2 \times \frac{16}{3} + 3 \times 2 = \frac{50}{3} = 16.67$

At $(4, 6)$: $P = 2 \times 4 + 3 \times 6 = 26$

At $(0, 8)$: $P = 2 \times 0 + 3 \times 8 = 24$

$\boxed{\text{Maximum } P = 26 \text{ at } (4, 6)}$

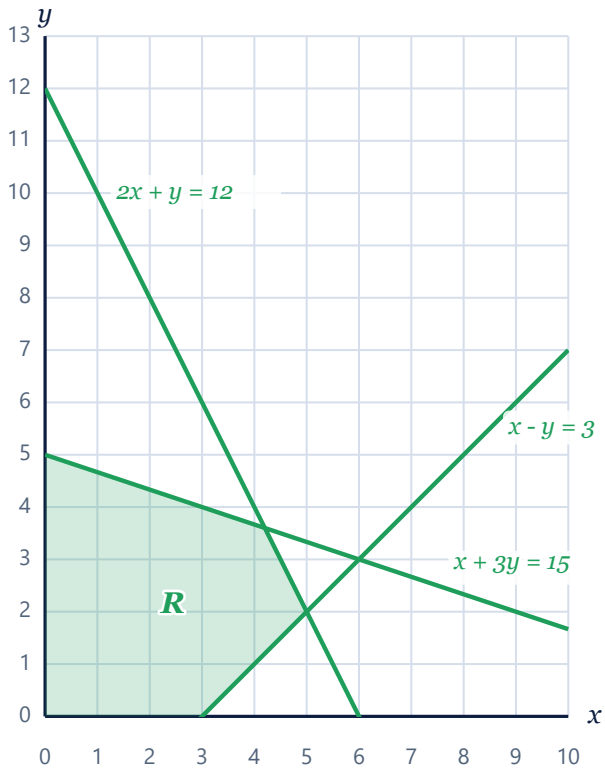
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4 A linear programming problem has the constraints

$2x + y \leq 12$, $x + 3y \leq 15$, $x - y \leq 3$, $x \geq 0$, $y \geq 0$

a. On the grid, draw lines to represent the constraints. Label the feasible region R.

b. Find the exact coordinates of every vertex of R.



(7 marks)

a)

Draw each line through two easy points

$2x + y = 12$ through $(0, 12)$ and $(6, 0)$

$x + 3y = 15$ through $(0, 5)$ and $(15, 0)$

$x - y = 3$ through $(3, 0)$ and $(8, 5)$

Test $(0, 0)$: $0 \leq 12$, $0 \leq 15$, $0 \leq 3$, so $(0, 0)$ is in R

Shade out the side of each line away from $(0, 0)$

R drawn and labelled

b)

Axes give $(0, 0)$ and $(0, 5)$

$x - y = 3$ meets $y = 0$ at $(3, 0)$

Solve $2x + y = 12$ (1) and $x - y = 3$ (2)

Eliminate x: (1) × 1, (2) × 2

$$2x + y = 12$$

$$2x - 2y = 6$$

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Subtract the second from the first

$3y = 6$

$y = 2$

Substitute into (1)

$2x = 12 - 1 \times 2$

$x = 5$

Solve $2x + y = 12$ (1) and $x + 3y = 15$ (2)

Eliminate x: (1) $\times 1$, (2) $\times 2$

$2x + y = 12$

$2x + 6y = 30$

Subtract the first from the second

$5y = 18$

$y = \frac{18}{5}$

Substitute into (1)

$2x = 12 - 1 \times \frac{18}{5}$

$x = \frac{21}{5}$

$(0, 0), (3, 0), (5, 2), (\frac{21}{5}, \frac{18}{5}), (0, 5)$

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5 A garden centre in Kent makes hanging baskets and window boxes to sell in spring.

- A hanging basket uses 3 litres of compost and a window box uses 5 litres. There are 150 litres of compost available.
- A hanging basket takes 20 minutes to plant and a window box takes 10 minutes. There are 8 hours of planting time available.
- At least 5 window boxes must be made.
- The number of hanging baskets must be no more than three times the number of window boxes.

The profit is £6 on each hanging basket and £4 on each window box. The garden centre wants to maximise its profit.

Formulate this as a linear programming problem. You should define your decision variables, state the objective and list the constraints as simplified inequalities.

(5 marks)

Define the decision variables

x = number of hanging baskets made

y = number of window boxes made

Objective

Maximise $P = 6x + 4y$

Constraints

Compost: $3x + 5y \leq 150$

Time: $20x + 10y \leq 8 \times 60 = 480$

Divide by 10: $2x + y \leq 48$

Window boxes: $y \geq 5$

Baskets: $x \leq 3y$

Non-negativity: $x \geq 0$

Max $P = 6x + 4y$; $3x + 5y \leq 150$, $2x + y \leq 48$, $y \geq 5$, $x \leq 3y$, $x \geq 0$

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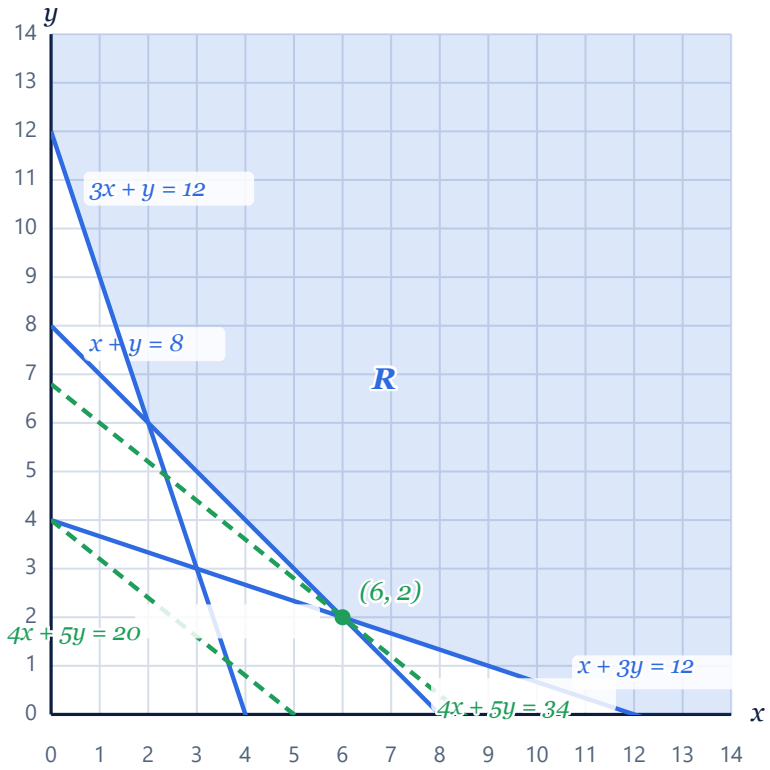
6 The diagram shows the feasible region, R, of a linear programming problem, defined by $x + y \geq 8$, $3x + y \geq 12$, $x + 3y \geq 12$, $x \geq 0$ and $y \geq 0$. R is unbounded.

The objective is to minimise $C = 4x + 5y$

a. Draw an objective line on the diagram and use it to identify the optimal vertex.

b. Find the coordinates of the optimal vertex and the minimum value of C.

c. Explain why the problem would have no solution if the objective were to maximise C.



(6 marks)

a)

Draw $4x + 5y = 20$ through $(0, 4)$ and $(5, 0)$

Gradient $-\frac{4}{5}$: shallower than $x + y = 8$ (gradient -1)

Slide the line parallel, away from O

It first touches R where $x + y = 8$ meets $x + 3y = 12$

Vertex on $x + y = 8$ and $x + 3y = 12$

b)

Solve $x + y = 8$ (1) and $x + 3y = 12$ (2)

Eliminate x: (1) × 1, (2) × 1

$$x + y = 8$$

$$x + 3y = 12$$

Subtract the first from the second

$$2y = 4$$

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$y = 2$

Substitute into (1)

$x = 8 - 1 \times 2$

$x = 6$

$C = 4 \times 6 + 5 \times 2 = 34$

$(6, 2)$; $C = 34$

c)

R is unbounded upwards and to the right

C can be made as large as we like

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7 A secondary school in Leeds is buying x laptops and y tablets.

- At least 40 devices must be bought in total.
- At most 70% of the devices may be tablets.
- At least a quarter of the devices must be laptops.
- A laptop costs £400 and a tablet costs £250. The budget is £18 000

Write down inequalities to model these constraints. Each must be simplified so that it has integer coefficients as small as possible.

(5 marks)

$\boxed{x + y \geqslant 40}$

Tablets at most 70% of the total

$y \leqslant 0.7(x + y)$

$y \leqslant 0.7x + 0.7y$

$0.3y \leqslant 0.7x$

Multiply by 10

$\boxed{3y \leqslant 7x}$

Laptops at least a quarter of the total

$x \geqslant \frac{1}{4}(x + y)$

$4x \geqslant x + y$

$\boxed{3x \geqslant y}$

Cost

$400x + 250y \leqslant 18000$

Divide by 50

$\boxed{8x + 5y \leqslant 360}$

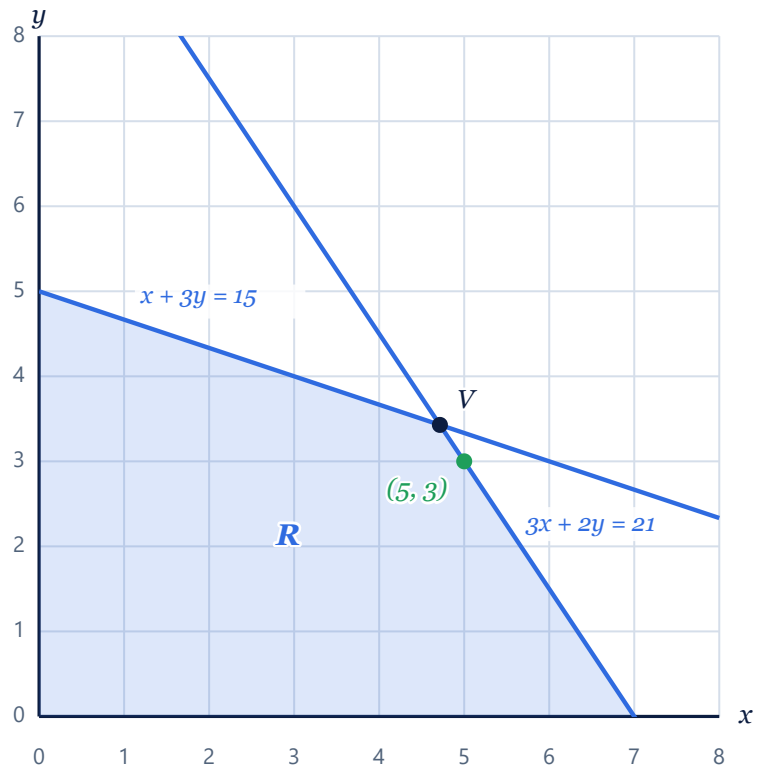
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8 A craft business makes x wooden toys and y jigsaws each week. The constraints are $3x + 2y \leq 21$, $x + 3y \leq 15$, $x \geq 0$ and $y \geq 0$, and the feasible region R is shown. The profit, in pounds, is $P = 5x + 4y$, to be maximised.

a. Find the exact coordinates of the vertex V and the value of P at V.

b. Show that V gives the maximum value of P over R.

c. Given that x and y must be integers, find the optimal number of toys and jigsaws and the maximum weekly profit.



(7 marks)

a)

Solve $3x + 2y = 21$ (1) and $x + 3y = 15$ (2)

Eliminate x: (1) × 1, (2) × 3

$$3x + 2y = 21$$

$$3x + 9y = 45$$

Subtract the first from the second

$$7y = 24$$

$$y = \frac{24}{7}$$

Substitute into (1)

$$3x = 21 - 2 \times \frac{24}{7}$$

$$x = \frac{33}{7}$$

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$$P = 5 \times \frac{33}{7} + 4 \times \frac{24}{7}$$

$$P = \frac{165}{7} + \frac{96}{7} = \frac{261}{7}$$

$V\left(\frac{33}{7}, \frac{24}{7}\right), P = \frac{261}{7} = 37.3$

b)

At $(0, 0)$: $P = 0$

At $(7, 0)$: $P = 35$

At $(0, 5)$: $P = 20$

ALL less than 37.3, so V is optimal

V gives the maximum

c)

Test integer points near V (and check the constraints)

$(4, 3)$: $18 \leq 21$, $13 \leq 15$, $P = 32$

$(5, 3)$: $21 \leq 21$, $14 \leq 15$, $P = 37$

$(4, 4)$: $4 + 12 = 16 > 15$, not feasible

$(5, 4)$: $15 + 8 = 23 > 21$, not feasible

$(6, 1)$: $20 \leq 21$, $9 \leq 15$, $P = 34$

$(7, 0)$: $P = 35$

No integer point can give 38 or more: $5x + 4y \leq 37.3$

5 toys, 3 jigsaws; £37

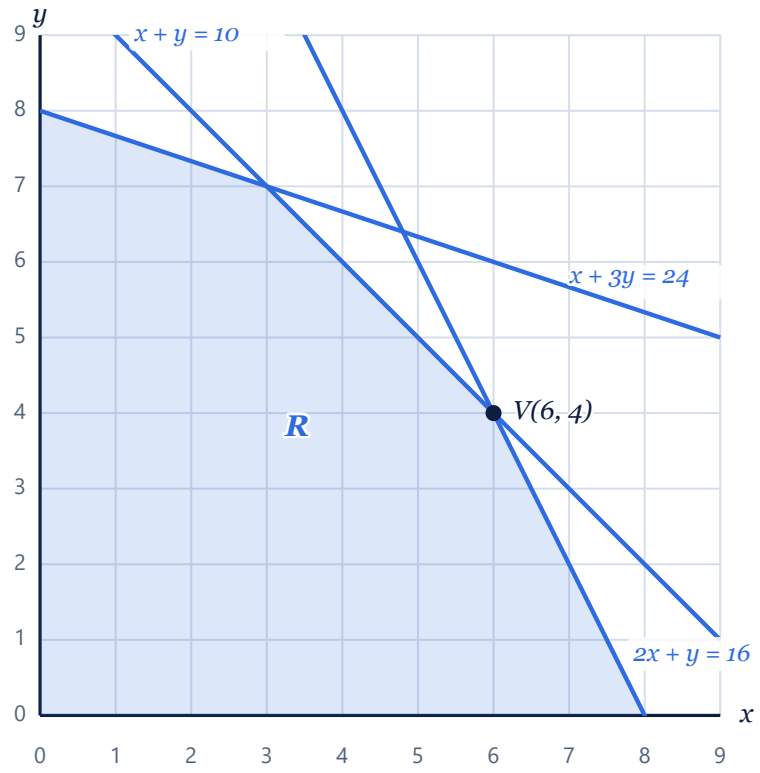
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9 The feasible region R is defined by $x + y \leq 10$, $2x + y \leq 16$, $x + 3y \leq 24$, $x \geq 0$ and $y \geq 0$, as shown.

a. Find the value of $P = 3x + 2y$ at each vertex of R, and hence state the optimal vertex when P is maximised.

The objective is changed to maximise $Q = x + ky$, where k is a positive constant. The vertex V(6, 4) is still optimal.

b. Find the range of possible values of k



(7 marks)

a)

Vertices: (0, 0), (8, 0), (0, 8)

$2x + y = 16$ and $x + y = 10$: subtract, $x = 6$, $y = 4$

$x + y = 10$ and $x + 3y = 24$: subtract, $2y = 14$, $y = 7$, $x = 3$

At (0, 0): $P = 0$

At (8, 0): $P = 24$

At (6, 4): $P = 18 + 8 = 26$

At (3, 7): $P = 9 + 14 = 23$

At (0, 8): $P = 16$

Optimal vertex (6, 4), $P = 26$

b)

Compare gradients at V

$2x + y = 16$ has gradient -2

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$x + y = 10$ has gradient -1

Objective line $x + ky = c$ has gradient $-\frac{1}{k}$

V stays optimal when the objective gradient lies between them

So $-2 \leqslant -\frac{1}{k} \leqslant -1$

$1 \leqslant \frac{1}{k} \leqslant 2$

k is positive, so take reciprocals and reverse

$$\boxed{\frac{1}{2} \leqslant k \leqslant 1}$$

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10 A furniture workshop in High Wycombe makes chairs and tables.

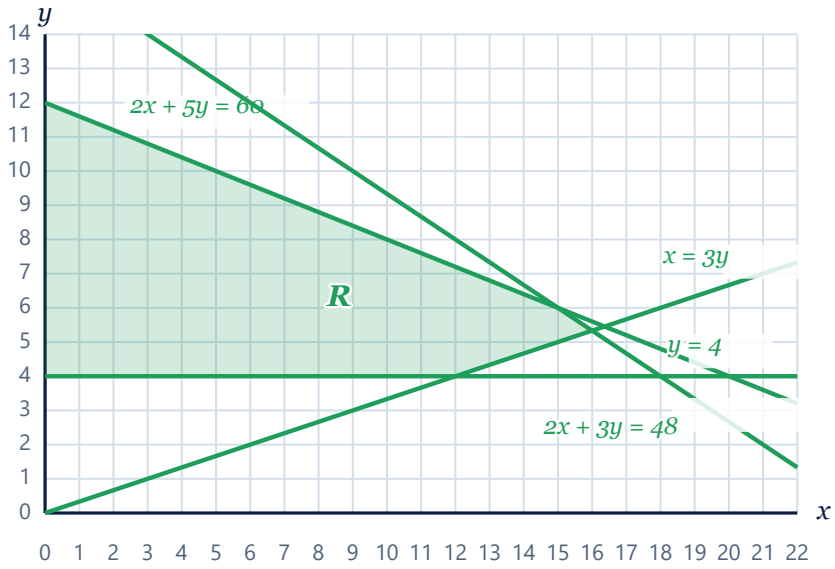
- A chair uses 2 m of timber and a table uses 5 m. There are 60 m of timber available each week.
- A chair takes 2 hours to make and a table takes 3 hours. There are 48 hours of labour available each week.
- At least 4 tables must be made each week.
- The number of chairs must be no more than three times the number of tables.

The profit is £30 on a chair and £55 on a table. Let x be the number of chairs and y the number of tables made each week.

a. Formulate the problem as a linear programming problem.

b. On the grid, draw the constraints and label the feasible region R.

c. Use the vertex method to find the number of chairs and tables that maximises the profit, and the maximum profit.



(10 marks)

a)

x = chairs, y = tables each week

Maximise $P = 30x + 55y$

Timber: $2x + 5y \leq 60$

Labour: $2x + 3y \leq 48$

Tables: $y \geq 4$

Chairs: $x \leq 3y$

$x \geq 0$

Max $P = 30x + 55y$; $2x + 5y \leq 60$, $2x + 3y \leq 48$, $y \geq 4$, $x \leq 3y$, $x \geq 0$

b)

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$2x + 5y = 60$ through $(0, 12)$ and $(30, 0)$

$2x + 3y = 48$ through $(0, 16)$ and $(24, 0)$

$y = 4$ is horizontal

$x = 3y$ through $(0, 0)$ and $(12, 4)$

R drawn and labelled

c)

Vertices of R

$x = 0$ with $y = 4$: $(0, 4)$

$x = 0$ with $2x + 5y = 60$: $(0, 12)$

Solve $y = 4$ (1) and $x - 3y = 0$ (2)

$$y = 4$$

$$x = 3 \times 4$$

$$x = 12$$

Solve $2x + 3y = 48$ (1) and $x - 3y = 0$ (2)

Eliminate x: (1) $\times 1$, (2) $\times 2$

$$2x + 3y = 48$$

$$2x - 6y = 0$$

Subtract the second from the first

$$9y = 48$$

$$y = \frac{16}{3}$$

Substitute into (1)

$$2x = 48 - 3 \times \frac{16}{3}$$

$$x = 16$$

Solve $2x + 5y = 60$ (1) and $2x + 3y = 48$ (2)

Subtract (2) from (1)

$$2y = 12$$

$$y = 6$$

Substitute into (1)

$$2x = 60 - 5 \times 6$$

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$x = 15$

Profit at each vertex

$P(0, 4) = 220$

$P(12, 4) = 580$

$P(16, 5.333) = 773.33$

$P(15, 6) = 780$

$P(0, 12) = 660$

15 chairs, 6 tables; £780

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11 A farmer in Lincolnshire mixes two feeds, A and B, for her sheep. She uses x kg of feed A and y kg of feed B each day. The mix must provide enough protein, fibre and minerals, which gives the constraints

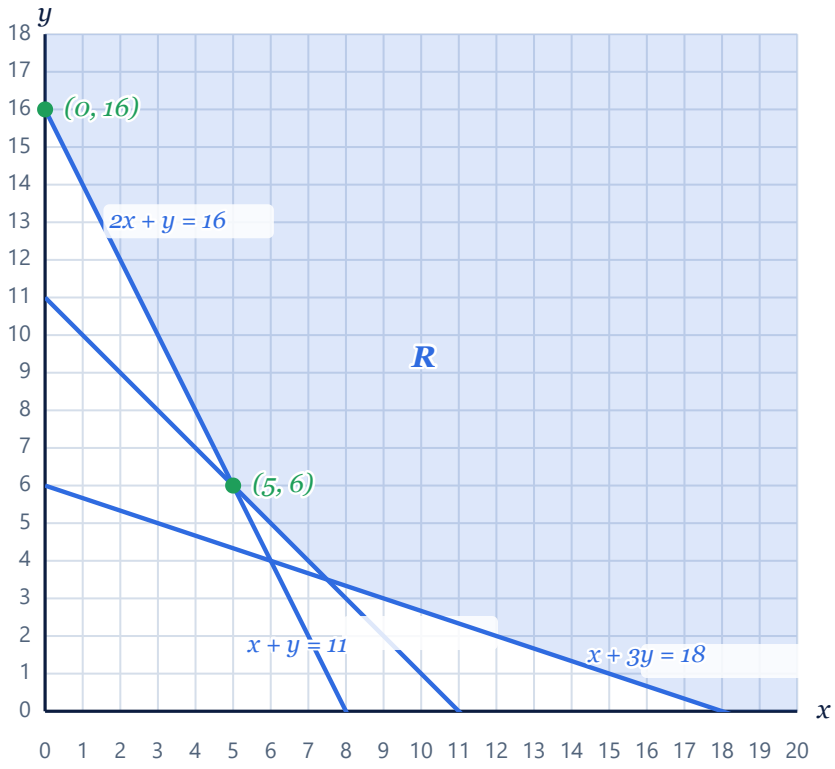
$2x + y \geq 16$, $x + y \geq 11$, $x + 3y \geq 18$, $x \geq 0$, $y \geq 0$

The feasible region R is shown. Feed A costs £3 per kg and feed B costs £2 per kg.

a. Find the exact coordinates of the vertices of R.

b. Find the amounts of each feed that minimise the daily cost, and the minimum cost.

c. The price of feed A rises to £5 per kg. Find the new minimum cost and the amounts of each feed.



(8 marks)

a)

$2x + y = 16$ meets the y-axis at $(0, 16)$

$x + 3y = 18$ meets the x-axis at $(18, 0)$

Solve $2x + y = 16$ (1) and $x + y = 11$ (2)

Eliminate x: (1) × 1, (2) × 2

$$2x + y = 16$$

$$2x + 2y = 22$$

Subtract the first from the second

$$y = 6$$

Substitute into (1)

$$2x = 16 - 1 \times 6$$

$$x = 5$$

Solve $x + y = 11$ (1) and $x + 3y = 18$ (2)

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Eliminate x: (1) × 1, (2) × 1

$$x + y = 11$$

$$x + 3y = 18$$

Subtract the first from the second

$$2y = 7$$

$$y = \frac{7}{2}$$

Substitute into (1)

$$x = 11 - 1 \times \frac{7}{2}$$

$$x = \frac{15}{2}$$

$\boxed{(0, 16), (5, 6), (7.5, 3.5), (18, 0)}$

b)

Evaluate $C = 3x + 2y$ at each vertex

$$C(0, 16) = 32$$

$$C(5, 6) = 15 + 12 = 27$$

$$C(7.5, 3.5) = 22.5 + 7 = 29.5$$

$$C(18, 0) = 54$$

$\boxed{\text{5 kg of A, 6 kg of B; £27}}$

c)

Evaluate $C = 5x + 2y$ at each vertex

$$C(0, 16) = 32$$

$$C(5, 6) = 25 + 12 = 37$$

$$C(7.5, 3.5) = 37.5 + 7 = 44.5$$

$$C(18, 0) = 90$$

$\boxed{\text{0 kg of A, 16 kg of B; £32}}$

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12 A sixth-form college in Bath is taking 300 students to a careers fair. It can hire minibuses, which seat 16 students and cost £120 each, and coaches, which seat 50 students and cost £400 each.

- There must be enough seats for all 300 students.
- Only 12 drivers are available, one for each vehicle.
- The college insists on at least as many minibuses as coaches.

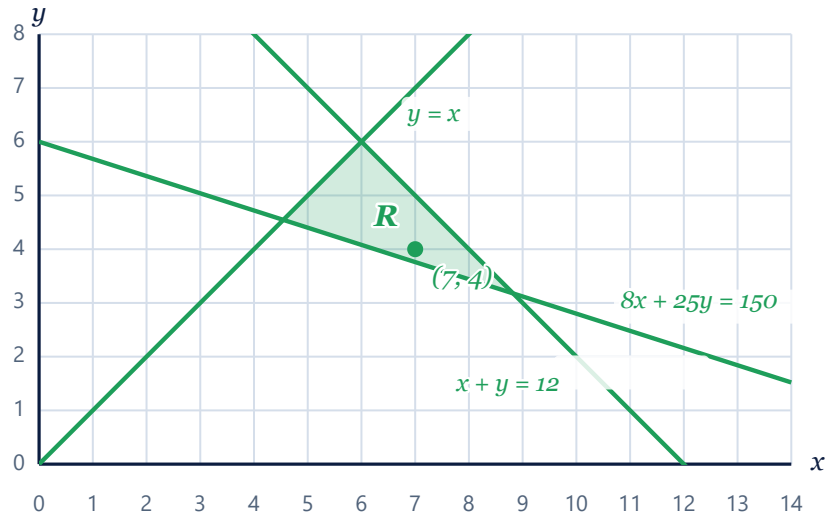
Let x be the number of minibuses and y the number of coaches.

a. Formulate the problem as a linear programming problem, simplifying your constraints.

b. On the grid, draw the constraints and label the feasible region R.

c. Find the vertices of R and the cost at each.

d. Find the number of minibuses and coaches that the college should hire, and the minimum cost.



(10 marks)

a)

x = minibuses, y = coaches

Minimise $C = 120x + 400y$

Seats: $16x + 50y \geqslant 300$

Divide by 2: $8x + 25y \geqslant 150$

Drivers: $x + y \leqslant 12$

Minibuses: $x \geqslant y$

$y \geqslant 0$

Min $C = 120x + 400y$; $8x + 25y \geqslant 150$, $x + y \leqslant 12$, $x \geqslant y$, $y \geqslant 0$

b)

$8x + 25y = 150$ through $(0, 6)$ and $(18.75, 0)$

$x + y = 12$ through $(0, 12)$ and $(12, 0)$

$y = x$ through $(0, 0)$ and $(6, 6)$

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R drawn and labelled

c)

Solve $8x + 25y = 150$ (1) and $x - y = 0$ (2)

Eliminate x: (1) × 1, (2) × 8

$$8x + 25y = 150$$
$$8x - 8y = 0$$

Subtract the second from the first

$$33y = 150$$
$$y = \frac{50}{11}$$

Substitute into (1)

$$8x = 150 - 25 \times \frac{50}{11}$$
$$x = \frac{50}{11}$$

Solve $8x + 25y = 150$ (1) and $x + y = 12$ (2)

Eliminate x: (1) × 1, (2) × 8

$$8x + 25y = 150$$
$$8x + 8y = 96$$

Subtract the second from the first

$$17y = 54$$
$$y = \frac{54}{17}$$

Substitute into (1)

$$8x = 150 - 25 \times \frac{54}{17}$$
$$x = \frac{150}{17}$$

Solve $x + y = 12$ (1) and $x - y = 0$ (2)

Eliminate x: (1) × 1, (2) × 1

$$x + y = 12$$
$$x - y = 0$$

Subtract the second from the first

$$2y = 12$$

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$y = 6$

Substitute into (1)

$$x = 12 - 1 \times 6$$

$$x = 6$$

$$C\text{ at }\left(\frac{50}{11}, \frac{50}{11}\right) = 520 \times \frac{50}{11} = 2363.64$$

$$C\text{ at }\left(\frac{150}{17}, \frac{54}{17}\right) = 120 \times \frac{150}{17} + 400 \times \frac{54}{17} = 2329.41$$

$$C\text{ at }(6, 6) = 720 + 2400 = 3120$$

$\left(\frac{50}{11}, \frac{50}{11}\right)$: £2363.64; $\left(\frac{150}{17}, \frac{54}{17}\right)$: £2329.41; (6, 6): £3120

d)

The optimal vertex is not an integer point

For each value of y, find the smallest x allowed

$y = 3$: $8x \geqslant 75$ gives $x \geqslant 10$, but $x + y \leqslant 12$ needs $x \leqslant 9$: impossible

($y = 0, 1, 2$ are also impossible)

$y = 4$: $8x \geqslant 50$ gives $x \geqslant 6.25$, so $x = 7$

$C(7, 4) = 840 + 1600 = 2440$

$y = 5$: $x \geqslant y$ gives $x = 5$

$C(5, 5) = 600 + 2000 = 2600$

$y = 6$: $x = 6$, $C = 3120$

Larger x for the same y only costs more

7 minibuses, 4 coaches; £2440

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