Minimum spanning trees

Worked solutions

Further Maths topic practice

This topic online

1 The network below has six vertices, A to F. The number on each arc is its weight.

a. Use Kruskal's algorithm to find a minimum spanning tree for the network. List the arcs in the order that you consider them and state whether each arc is added to the tree.

b. State the weight of the minimum spanning tree.

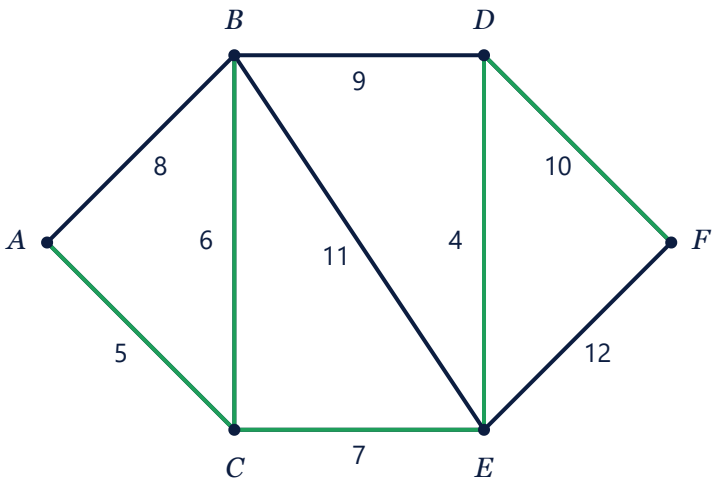


Diagram NOT accurately drawn

(4 marks)

List the arcs in order of weight

OE 4, AC 5, BC 6, CE 7, AB 8, BD 9, DF 10, BE 11, EF 12

Take each arc in turn: add it unless it makes a cycle

OE (4): add

AC (5): add

BC (6): add

CE (7): add

AB (8): reject, makes a cycle

BD (9): reject, makes a cycle

DF (10): add

6 vertices, so the tree has 5 arcs: stop

Add the weights of the arcs in the tree

Weight = 4 + 5 + 6 + 7 + 10 = 32

Tree OE, AC, BC, CE, DF; weight 32

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

2 The network below shows the lengths, in metres, of paths between seven huts, A to G, at a scout camp in the Lake District.

a. Use Prim's algorithm, starting at G, to find a minimum spanning tree for the network. State the order in which you select the arcs.

b. State the total length of the minimum spanning tree.

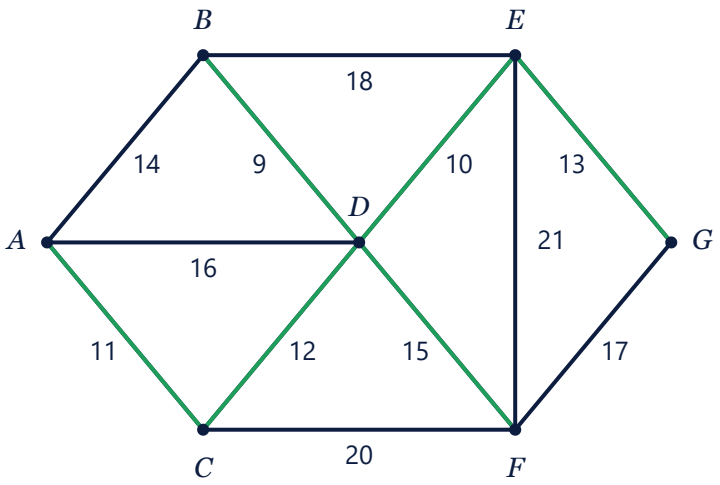


Diagram NOT accurately drawn

(4 marks)

Start at G; each time add the shortest arc joining a new vertex to the tree

GE (13): add E

EO (10): add O

OB (9): add B

OC (12): add C

CA (11): add A

OF (15): add F

Weight = 13 + 10 + 9 + 12 + 11 + 15 = 70

GE, EO, OB, OC, CA, OF; weight 70

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

3 The table shows the distances, in miles, between six villages in Shropshire, A to F.

	A	B	C	D	E	F
A	–	23	31	18	27	35
B	23	–	16	29	21	30
C	31	16	–	25	14	22
D	18	29	25	–	26	33
E	27	21	14	26	–	19
F	35	30	22	33	19	–

a. Use Prim's algorithm, starting at A, to find a minimum spanning tree for the six villages. You must state the order in which you select the arcs.

b. State the weight of the minimum spanning tree.

(4 marks)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 18 in column A, row D: add AD

Delete row D, number column D as 2

Smallest is 23 in column A, row B: add AB

Delete row B, number column B as 3

Smallest is 16 in column B, row C: add BC

Delete row C, number column C as 4

Smallest is 14 in column C, row E: add CE

Delete row E, number column E as 5

Smallest is 19 in column E, row F: add EF

Delete row F, number column F as 6

Weight = 18 + 23 + 16 + 14 + 19 = 90

AD, AB, BC, CE, EF; weight 90

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

4 The table shows the lengths, in km, of the roads between seven towns, A to G, in Lincolnshire. A dash means there is no direct road.

	A	B	C	D	E	F	G
A	–	32	–	28	25	–	–
B	32	–	30	–	22	–	–
C	–	30	–	–	27	34	–
D	28	–	–	–	24	–	36
E	25	22	27	24	–	29	31
F	–	–	34	–	29	–	26
G	–	–	–	36	31	26	–

a. The seven towns are shown in the diagram. Draw the network of roads, labelling each arc with its length.

b. Use Kruskal's algorithm to find a minimum spanning tree. List the arcs in the order that you consider them and state whether each is added.

c. State the total length of your minimum spanning tree.

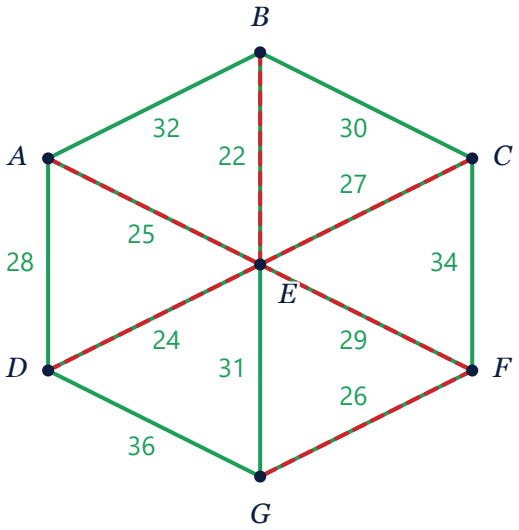


Diagram NOT accurately drawn

(6 marks)

a)

Each number in the table is an arc; a dash means no direct road

Arcs: AB 32, AD 28, AE 25, BC 30, BE 22, CE 27, CF 34, DE 24, DG 36, EF 29, EG 31, FG 26

Network drawn with 12 arcs

b)

List the arcs in order of weight

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

BE 22, OE 24, AE 25, FG 26, CE 27, AO 28, EF 29, BC 30, EG 31, AB 32, CF 34, OG 36

Take each arc in turn: add it unless it makes a cycle

BE (22): add

OE (24): add

AE (25): add

FG (26): add

CE (27): add

AO (28): reject, makes a cycle

EF (29): add

7 vertices, so the tree has 6 arcs: stop

c)

Weight = 22 + 24 + 25 + 26 + 27 + 29 = 153

153 km

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

5 The table shows the times, in minutes, to drive between seven depots, A to G, of a delivery firm in the West Midlands.

	A	B	C	D	E	F	G
A	–	42	57	36	61	48	70
B	42	–	33	51	45	39	64
C	57	33	–	47	29	55	41
D	36	51	47	–	53	31	66
E	61	45	29	53	–	50	38
F	48	39	55	31	50	–	58
G	70	64	41	66	38	58	–

a. Use Prim's algorithm, starting at D, to find a minimum spanning tree. State the order in which you add the vertices to the tree.

b. Find the weight of the tree.

c. Kruskal's algorithm is also applied to the same table. Explain whether Kruskal's algorithm could give a different minimum spanning tree.

(6 marks)

a)

Start at D: delete row D, number column D as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 31 in column D, row F: add DF

Delete row F, number column F as 2

Smallest is 36 in column D, row A: add DA

Delete row A, number column A as 3

Smallest is 39 in column F, row B: add FB

Delete row B, number column B as 4

Smallest is 33 in column B, row C: add BC

Delete row C, number column C as 5

Smallest is 29 in column C, row E: add CE

Delete row E, number column E as 6

Smallest is 38 in column E, row G: add EG

Delete row G, number column G as 7

Order: D, F, A, B, C, E, G

b)

Weight = 31 + 36 + 39 + 33 + 29 + 38 = 206

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

206 minutes

c)

Check for equal weights

ALL 21 entries are different

So only one spanning tree has the least weight

Kruskal may pick the arcs in a different order but gives the same tree

No, the same tree

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

6 The network below shows the costs, in thousands of pounds, of laying water pipes between six farms, A to F, in Devon.

a. Use Kruskal's algorithm to find a minimum spanning tree. List the arcs in the order that you consider them and state whether each is added.

b. State the minimum cost.

c. Find the number of different minimum spanning trees for this network. Give a reason for your answer.

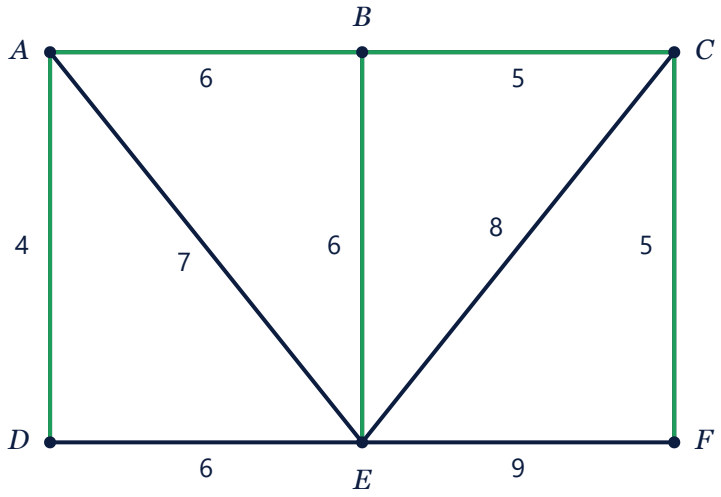


Diagram NOT accurately drawn

(5 marks)

a)

List the arcs in order of weight

AD 4, BC 5, CF 5, AB 6, BE 6, DE 6, AE 7, CE 8, EF 9

AD (4): add

BC (5): add

CF (5): add

AB (6): add

BE (6): add

DE (6): reject, makes the cycle ABED

6 vertices, so 5 arcs: stop

AD, BC, CF, AB, BE

b)

Cost = 4 + 5 + 5 + 6 + 6 = 26

£26000

c)

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

AD, BC, CF are always in the tree

Then two of AB, BE, DE (all weight 6) are needed

All three together make the cycle ABED, any two do not

Number of choices = 3

Using AE (7) instead would cost more

3 minimum spanning trees

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

7 The network below shows the lengths, in metres, of cable needed to link eight buildings, A to H, on a university campus in Leeds.

a. Use Prim's algorithm, starting at C, to find a minimum spanning tree. State the order in which you select the arcs.

b. Kruskal's algorithm is applied to the same network. State the order in which the arcs are added to the tree.

c. State the total length of cable needed.

d. Explain why Kruskal's algorithm needs a check for cycles at every step but Prim's algorithm does not.

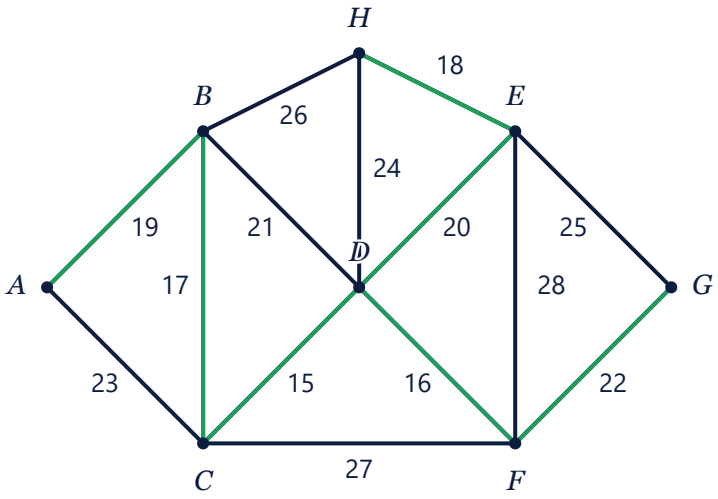


Diagram NOT accurately drawn

(7 marks)

a)

Start at C; each time add the shortest arc joining a new vertex to the tree

CO (15): add O

OF (16): add F

CB (17): add B

BA (19): add A

OE (20): add E

EH (18): add H

FG (22): add G

CO, OF, CB, BA, OE, EH, FG

b)

List the arcs in order of weight

CO 15, OF 16, BC 17, EH 18, AB 19, OE 20, BO 21, FG 22, AC 23, OH 24, EG 25, BH 26, CF 27, EF 28

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

Take each arc in turn: add it unless it makes a cycle

CO (15): add

OF (16): add

BC (17): add

EH (18): add

AB (19): add

OE (20): add

BO (21): reject, makes a cycle

FG (22): add

8 vertices, so the tree has 7 arcs: stop

CO, OF, BC, EH, AB, OE, FG

c)

Length = 15 + 16 + 17 + 18 + 19 + 20 + 22 = 127

127 m

d)

Prim: each new arc joins the tree to a vertex not yet in it so a cycle can never be made

Kruskal: arcs are taken in size order from anywhere an arc can join two vertices already linked, making a cycle

Prim only adds new vertices; Kruskal must check

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

8 The network below has six vertices. The weight of arc BD is x, where $x > 0$

a. Given that $x = 15$, use Kruskal's algorithm to find a minimum spanning tree and state its weight.

b. Find the range of values of x for which arc BD is in every minimum spanning tree of the network.

c. For these values, write down the weight of the minimum spanning tree in terms of x

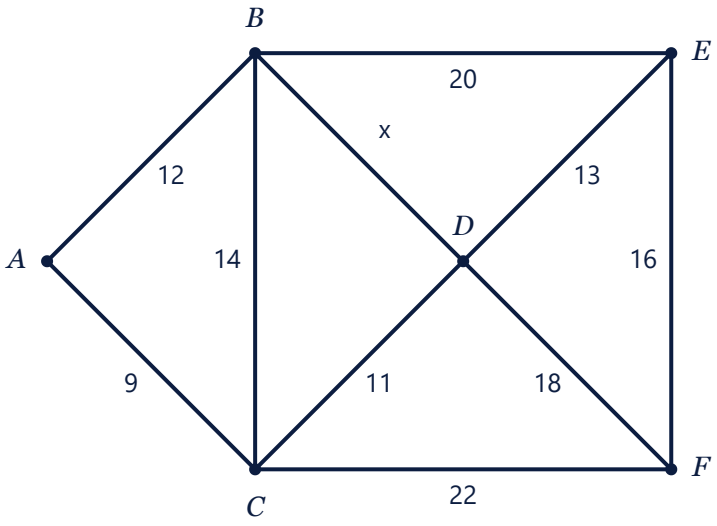


Diagram NOT accurately drawn

(6 marks)

a)

List the arcs in order of weight (BD = 15)

AC 9, CD 11, AB 12, DE 13, BC 14, BD 15, EF 16, DF 18, BE 20, CF 22

AC (9): add

CD (11): add

AB (12): add

DE (13): add

BC (14): reject, makes a cycle

BD (15): reject, makes a cycle

EF (16): add

5 arcs for 6 vertices: stop

Weight = 9 + 11 + 12 + 13 + 16 = 61

AC, CD, AB, DE, EF; 61

b)

Kruskal meets BD before the arc it would replace

Path from B to D in the tree: B-A-C-D

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

Arcs on the path: AB 12, AC 9, CD 11

BD is added before AB is reached when $x < 12$

then AB is rejected (cycle ABDC)

If $x = 12$, AB or BD could be chosen, so BD is not in every tree

$$0 < x < 12$$

c)

BD replaces AB

Weight $= 61 - 12 + x$

$$49 + x$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

9 A cycle club in Cornwall wants to resurface tracks so that seven checkpoints, P to V, are all linked. The network shows the lengths, in km, of the possible tracks.

a. Use Kruskal's algorithm to find a minimum spanning tree. List the arcs in the order that you consider them and state whether each is added.

b. State the total length of track in the tree.

The landowner insists that the track QS must be resurfaced.

c. Find the minimum total length of track that links all seven checkpoints and includes QS, stating the arcs used.

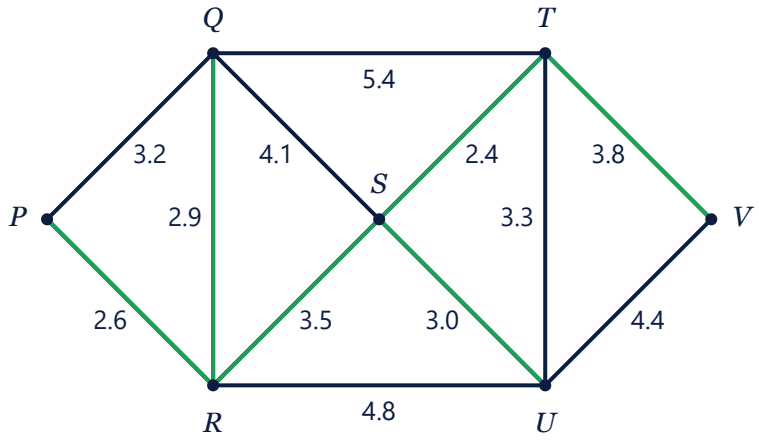


Diagram NOT accurately drawn

(7 marks)

a)

List the arcs in order of weight

ST 2.4, PR 2.6, QR 2.9, SU 3, PQ 3.2, TU 3.3, RS 3.5, TV 3.8, QS 4.1, UV 4.4, RU 4.8, QT 5.4

Take each arc in turn: add it unless it makes a cycle

ST (2.4): add

PR (2.6): add

QR (2.9): add

SU (3): add

PQ (3.2): reject, makes a cycle

TU (3.3): reject, makes a cycle

RS (3.5): add

TV (3.8): add

7 vertices, so the tree has 6 arcs: stop

ST, PR, QR, SU, RS, TV

b)

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

Length = 2.4 + 2.6 + 2.9 + 3 + 3.5 + 3.8 = 18.2

18.2 km

c)

Put QS in the tree first, then carry on with Kruskal

QS (4.1): add first

ST (2.4): add

PR (2.6): add

QR (2.9): add

SU (3): add

PQ (3.2): reject, makes a cycle

TU (3.3): reject, makes a cycle

RS (3.5): reject, makes a cycle

TV (3.8): add

Length = 4.1 + 2.4 + 2.6 + 2.9 + 3 + 3.8 = 18.8

QS, ST, PR, QR, SU, TV; 18.8 km

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

10 A broadband company plans to link six villages in Leicestershire with fibre cable laid along roads: Ashby (A), Brill (B), Cotes (C), Dunton (D), Elm (E) and Foxley (F). The table shows the road distances, in km, between the villages.

	A	B	C	D	E	F
A	–	6.4	8.1	5.2	9.7	7.3
B	6.4	–	4.6	7.8	5.9	8.8
C	8.1	4.6	–	6.9	3.8	9.1
D	5.2	7.8	6.9	–	8.4	4.9
E	9.7	5.9	3.8	8.4	–	7.6
F	7.3	8.8	9.1	4.9	7.6	–

a. Use Prim's algorithm, starting at Ashby, to find the minimum length of cable needed to link all six villages. You must state the order in which you select the arcs.

b. Laying the cable costs £18 500 per km. Find the minimum cost of the network.

c. Give one practical reason why the company might not use the network found in part (a). (7 marks)

a)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 5.2 in column A, row D: add AD

Delete row D, number column D as 2

Smallest is 4.9 in column D, row F: add DF

Delete row F, number column F as 3

Smallest is 6.4 in column A, row B: add AB

Delete row B, number column B as 4

Smallest is 4.6 in column B, row C: add BC

Delete row C, number column C as 5

Smallest is 3.8 in column C, row E: add CE

Delete row E, number column E as 6

Length = 5.2 + 4.9 + 6.4 + 4.6 + 3.8 = 24.9

AD, DF, AB, BC, CE; 24.9 km

b)

Cost = 24.9 × 18 500

Cost = 460650

£460650

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

c)

A spanning tree has only one route between villages

one damaged cable cuts off every village beyond it

No back-up route if a cable fails

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

11 The table shows the distances, in metres, between seven classroom blocks, A to G, in a large secondary school in Bristol. The school wants to link every block with a covered walkway of minimum total length.

	A	B	C	D	E	F	G
A	–	18	25	31	22	37	29
B	18	–	14	27	30	33	24
C	25	14	–	16	28	26	35
D	31	27	16	–	19	21	34
E	22	30	28	19	–	32	17
F	37	33	26	21	32	–	23
G	29	24	35	34	17	23	–

a. Use Prim's algorithm, starting at A, to find a minimum spanning tree. State the order in which you select the arcs.

b. State the length of the minimum spanning tree.

A new sports hall, H, is built. Its distances, in metres, to the other blocks are

	A	B	C	D	E	F	G
H	20	26	15	12	27	13	30

c. Find a minimum spanning tree for all eight buildings, explaining your method, and state its length.

(8 marks)

a)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 18 in column A, row B: add AB

Delete row B, number column B as 2

Smallest is 14 in column B, row C: add BC

Delete row C, number column C as 3

Smallest is 16 in column C, row D: add CD

Delete row D, number column D as 4

Smallest is 19 in column D, row E: add DE

Delete row E, number column E as 5

Smallest is 17 in column E, row G: add EG

Delete row G, number column G as 6

Smallest is 21 in column D, row F: add DF

Delete row F, number column F as 7

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

AB, BC, CO, OE, EG, OF

b)

Length = 18 + 14 + 16 + 19 + 17 + 21 = 105

105 m

c)

The new tree only needs the old tree arcs and the arcs to H

Old tree arcs and H arcs in order of weight:

OH 12, FH 13, BC 14, CH 15, CO 16, EG 17, AB 18, OE 19, AH 20, OF 21, BH 26, EH 27, GH 30

OH (12): add

FH (13): add

BC (14): add

CH (15): add

CO (16): reject, makes a cycle

EG (17): add

AB (18): add

OE (19): add

Length = 12 + 13 + 14 + 15 + 17 + 18 + 19 = 108

OH, FH, BC, CH, EG, AB, OE; 108 m

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

12 The table shows the distances, in km, between six petrol stations, A to F, on a motorway network. The distance from C to D is k km.

	A	B	C	D	E	F
A	–	30	41	26	38	45
B	30	–	24	33	29	40
C	41	24	–	k	35	27
D	26	33	k	–	36	39
E	38	29	35	36	–	31
F	45	40	27	39	31	–

Prim's algorithm, starting at A, is applied to the table. The vertices are added to the tree in the order A, D, C, B, F, E.

a. Find the range of possible values of k

b. Write down, in terms of k, the weight of the minimum spanning tree.

c. Given instead that $k = 34$, find the order in which the vertices are added and the weight of the minimum spanning tree.

(7 marks)

a)

Delete row A, number column A

Column A: B 30, C 41, D 26, E 38, F 45

Smallest is 26: add AD

Now look in columns A and D

B: smallest of 30, 33 is 30

C: smallest of 41, k

E: smallest of 38, 36 is 36

F: smallest of 45, 39 is 39

C is added next, so k must be the smallest

$k < 30$

k is a distance, so $k > 0$

Check the rest of the order (columns A, D, C)

B: 24 (BC), E: 35 (CE), F: 27 (CF)

Add CB (24)

Then E: 29 (BE), F: 27 (CF): add CF (27)

Then E: 29 (BE): add BE (29)

This order holds for every $k < 30$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$0 < k < 30$

b)

Weight = $26 + k + 24 + 27 + 29$

Weight = $106 + k$

$106 + k$

c)

Columns A and D: B 30, C 34, E 36, F 39

Add AB (30)

Columns A, D, B: C 24, E 29, F 39: add BC (24)

Columns A, D, B, C: E 29, F 27: add CF (27)

Last: E 29: add BE (29)

Weight = $26 + 30 + 24 + 27 + 29 = 136$

A, D, B, C, F, E; 136 km

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.