Planarity algorithm

Worked solutions

Further Maths topic practice

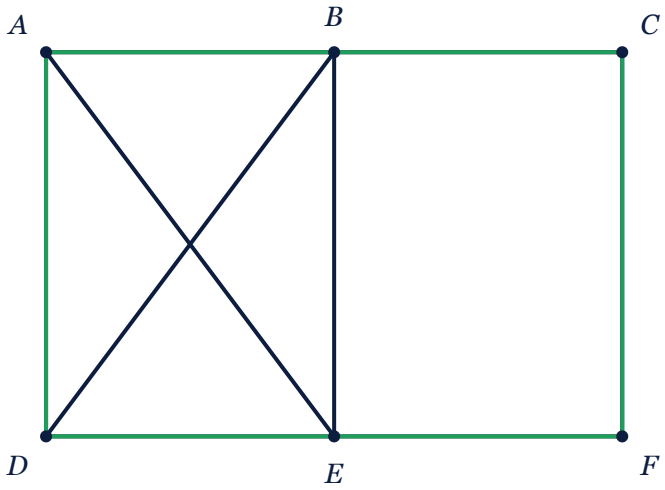
This topic online

1 The graph G is shown below.

a. Explain what is meant by a planar graph.

b. Explain what is meant by a Hamiltonian cycle.

c. Write down a Hamiltonian cycle for G, starting and finishing at A.



(4 marks)

a)

A graph that can be drawn in a plane

so that no two arcs meet except at a vertex

b)

A cycle that passes through every vertex exactly once

and returns to the starting vertex

c)

Go round the outside of the drawing

AB, BC, CF, FE, ED, DA are all arcs of G

Each of the six vertices is visited once

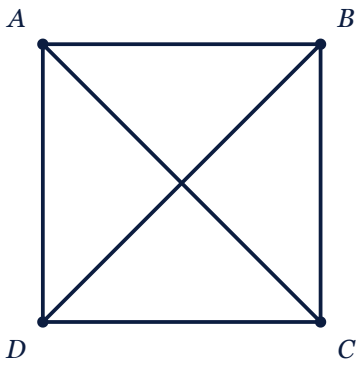
A B C F E D A

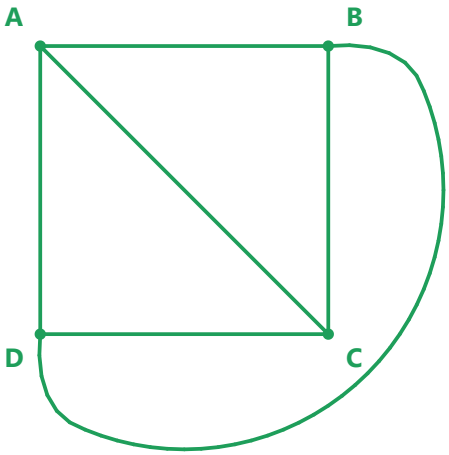
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2 The graph below has four vertices and six arcs. In the drawing, arcs AC and BD cross.

Taking ABCDA as the Hamiltonian cycle, use the planarity algorithm to show that the graph is planar.

Draw the graph without any arcs crossing.





(3 marks)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D A

Arcs not in the cycle: AC, BD

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AC: I

BD crosses AC (I), so BD: O

Inside: AC

Outside: BD

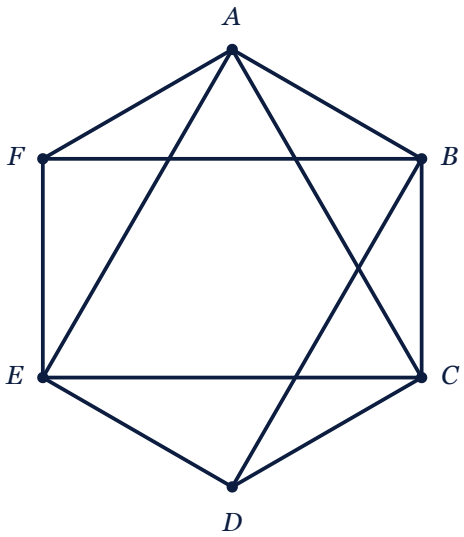
No two inside arcs cross and no two outside arcs cross

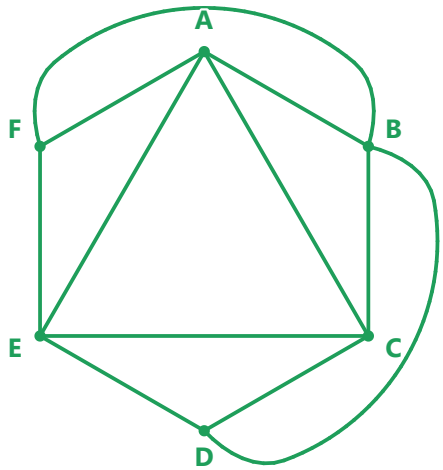
Planar: AC inside the square, BD drawn round the outside

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3 The graph below has six vertices, A to F.

Taking ABCDEFA as the Hamiltonian cycle, use the planarity algorithm to determine whether the graph is planar. You must make your working clear and justify your answer.





(4 marks)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AC, AE, BD, BF, CE

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AC: I

BD crosses AC (I), so BD: O

BF crosses AC (I), so BF: O

CE crosses BD (O), so CE: I

AE crosses BF (O), so AE: I

Inside: AC, CE, AE

Outside: BD, BF

No two inside arcs cross and no two outside arcs cross

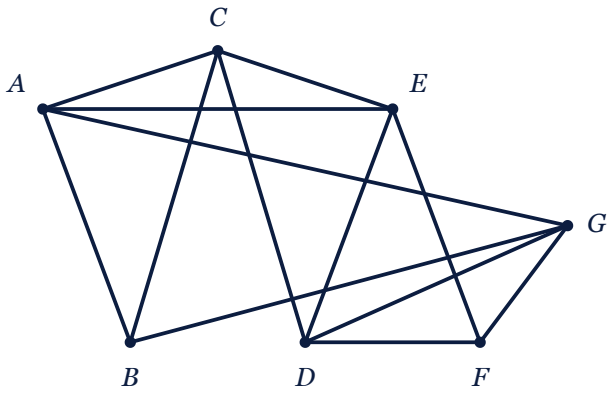
The graph is planar (drawn without crossings)

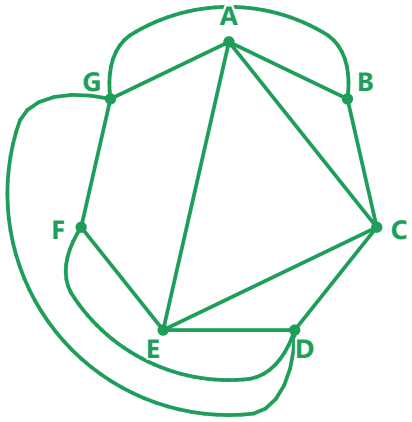
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4 The graph below has seven vertices, A to G, and thirteen arcs.

a. Starting at A, find a Hamiltonian cycle for the graph.

b. Use your Hamiltonian cycle and the planarity algorithm to determine whether the graph is planar. You must make your working clear and justify your answer.





(5 marks)

a)

A to B to C to D to E to F to G, and GA is an arc

A B C D E F G A

b)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F G A

Arcs not in the cycle: AC, AE, BG, CE, DF, DG

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AC: I

BG crosses AC (I), so BG: O

AE crosses BG (O), so AE: I

DF crosses AE (I), so DF: O

DG crosses AE (I), so DG: O

CE crosses DF (O), so CE: I

Inside: AC, AE, CE

Outside: BG, DF, DG

No two inside arcs cross and no two outside arcs cross

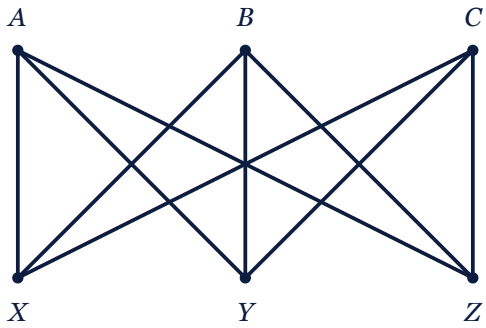
The graph is planar (drawn without crossings)

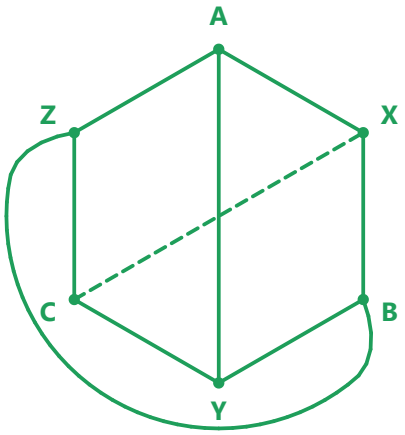
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5 Three new houses, A, B and C, are each to be connected by an underground pipe to each of three supply points: water X, gas Y and a heat pump Z. The pipes must all be laid at the same depth, so no two pipes may cross. The graph below shows the nine connections.

a. Write down a Hamiltonian cycle for the graph, starting at A.

b. Use the planarity algorithm to determine whether the pipes can be laid without crossing. You must make your working clear.





(5 marks)

a)

Each arc joins a house to a supply point, so the cycle alternates

A X B Y C Z A

b)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A X B Y C Z A

Arcs not in the cycle: AY, BZ, CX

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AY: I

BZ crosses AY (I), so BZ: O

CX crosses AY (I), so CX: O

CX also crosses BZ, which is outside

CX is outside and so is BZ, and they cross

CX can be placed neither inside nor outside without a crossing

Not planar: the pipes cannot all be laid without crossing

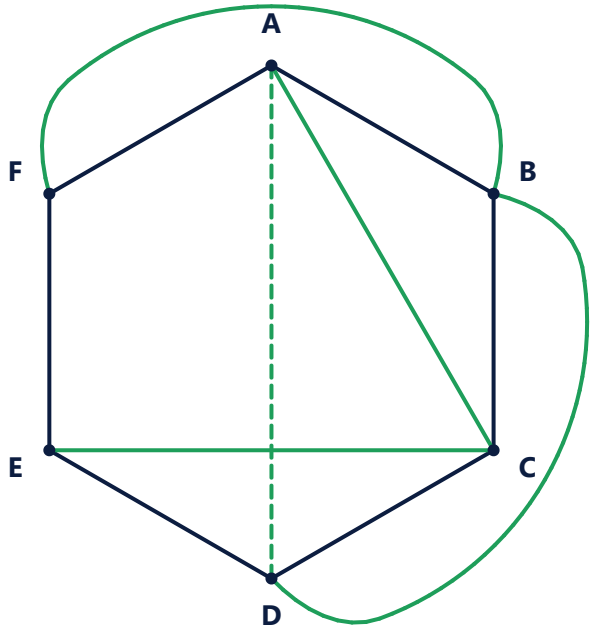
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6 A graph G has six vertices, A to F. The vertices adjacent to each vertex are listed below.

Vertex	Adjacent to
A	B, C, D, F
B	A, C, D, F
C	A, B, D, E
D	A, B, C, E, F
E	C, D, F
F	A, B, D, E

a. State the number of arcs in G.

b. Taking ABCDEFA as the Hamiltonian cycle, which is drawn below, use the planarity algorithm to determine whether G is planar. You must make your working clear and justify your answer.



(5 marks)

a)

sum of degrees = 4 + 4 + 4 + 5 + 3 + 4 = 24

arcs = 24 ÷ 2 = 12

12 arcs

b)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AC, AD, BD, BF, CE, DF

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

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AC: I

BO crosses AC (I), so BO: O

BF crosses AC (I), so BF: O

CE crosses BO (O), so CE: I

AO crosses BF (O), so AO: I

AO also crosses CE, which is inside

AO is inside and so is CE, and they cross

AO can be placed neither inside nor outside without a crossing

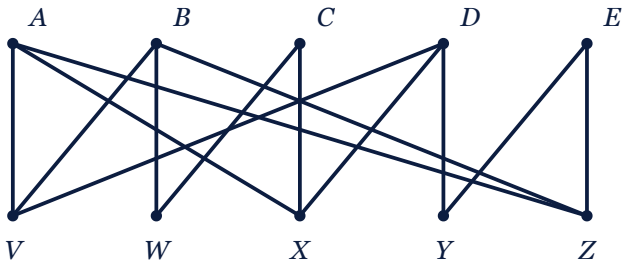
G is not planar

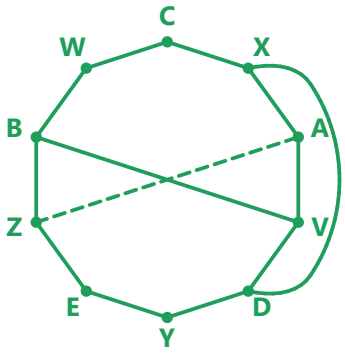
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7 The graph below has ten vertices and thirteen arcs. A Hamiltonian cycle for the graph begins C, X, A, V, ...

a. Complete the Hamiltonian cycle.

b. Hence use the planarity algorithm to determine whether the graph is planar. You must make your working clear and justify your answer.





(5 marks)

a)

Each vertex must be used once; at each step take the only unused neighbour that keeps the cycle possible

V: neighbours A, B, O; A used, and B is needed later to reach W, so V to O

O: neighbours X, Y, V; X and V used, so O to Y

Y: neighbours O, E; O used, so Y to E

E: neighbours Y, Z; Y used, so E to Z

Z: neighbours A, B, E; A and E used, so Z to B

B: neighbours V, W, Z; V and Z used, so B to W

W: neighbours B, C; B used, so W to C, which closes the cycle

C X A V O Y E Z B W C

b)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: C X A V O Y E Z B W C

Arcs not in the cycle: VB, XO, ZA

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

VB: I

XO crosses VB (I), so XO: O

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ZA crosses VB (I), so ZA: O

ZA also crosses XO, which is outside

ZA is outside and so is XO, and they cross

ZA can be placed neither inside nor outside without a crossing

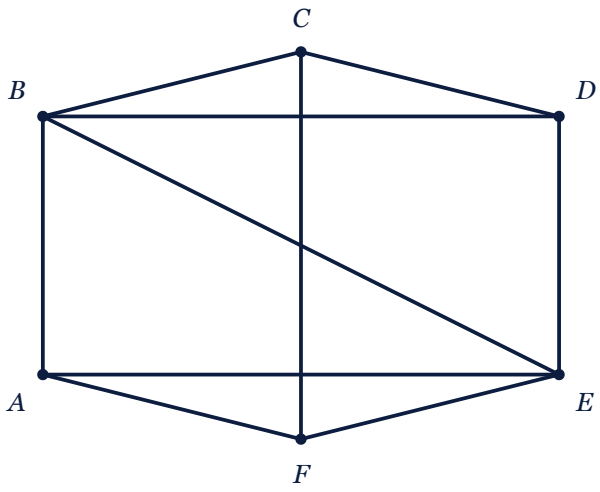
The graph is not planar

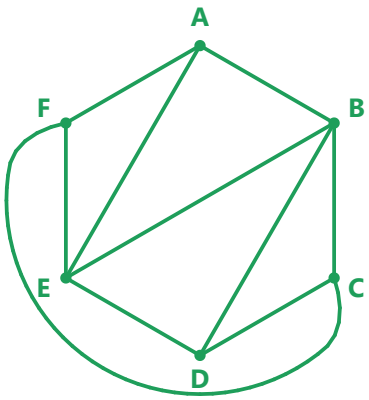
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8 The graph below shows the footpaths, drawn as arcs, between six picnic areas, A to F, in a park.

a. Taking ABCDEFA as the Hamiltonian cycle, use the planarity algorithm to show that the graph is planar.

b. A new footpath AD is to be added. Determine whether the graph remains planar. You must make your working clear.





(7 marks)

a)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AE, BD, BE, CF

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AE: I

CF crosses AE (I), so CF: O

BD crosses CF (O), so BD: I

BE crosses CF (O), so BE: I

Inside: AE, BD, BE

Outside: CF

No two inside arcs cross and no two outside arcs cross

The graph is planar (drawn without crossings)

b)

Run the algorithm again with AD included

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AE, BD, BE, CF, AD

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Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AE: I

CF crosses AE (I), so CF: O

BO crosses CF (O), so BO: I

BE crosses CF (O), so BE: I

AO crosses CF (O), so AO: I

AO also crosses BE, which is inside

AO is inside and so is BE, and they cross

AO can be placed neither inside nor outside without a crossing

With AO added the graph is not planar

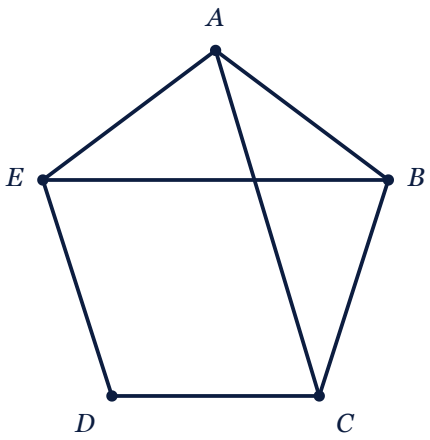
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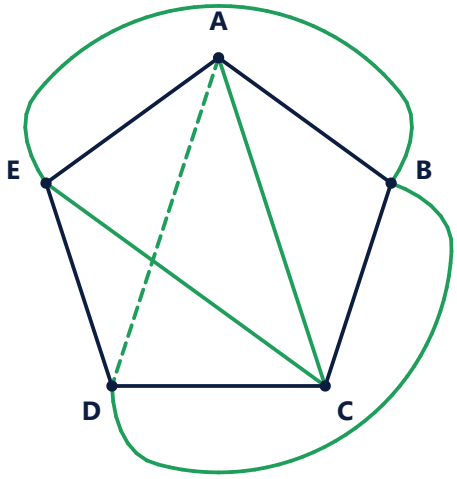
9 The graph G below has five vertices and seven arcs.

a. State the number of arcs in the complete graph K_n, in terms of n.

b. Find the number of arcs that would need to be added to G to make the complete graph K_5, and list them.

c. Taking ABCDEA as the Hamiltonian cycle, which is drawn on the right below, use the planarity algorithm to show that K_5 is not planar.





(6 marks)

a)

Each vertex joins to n - 1 others; each arc counted twice

$n(n-1) \div 2$

b)

K5: 5 × 4 ÷ 2 = 10 arcs

10 − 7 = 3

Pairs not joined in G: AD, BD, CE

3 arcs: AD, BD, CE

c)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E A

Arcs not in the cycle: AC, AD, BD, BE, CE

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AC: I

BD crosses AC (I), so BD: O

BE crosses AC (I), so BE: O

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CE crosses BD (O), so CE: I

AD crosses BE (O), so AD: I

AD also crosses CE, which is inside

AD is inside and so is CE, and they cross

AD can be placed neither inside nor outside without a crossing

K5 is not planar

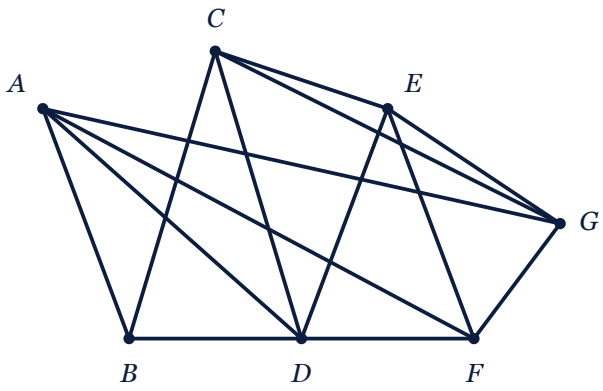
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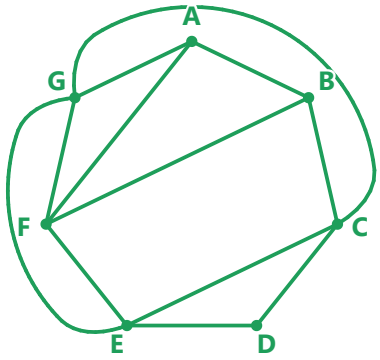
10 A circuit board has seven components, A to G. The graph below shows the thirteen copper tracks that must join pairs of components. Tracks printed on one side of the board must not cross.

a. Show that A B C D E F G A is a Hamiltonian cycle for the graph.

b. Use the planarity algorithm to determine whether all thirteen tracks can be printed on one side of the board.

c. The designer moves one track to the other side of the board. Find a track that could be moved so that the remaining twelve tracks can be printed on one side without crossing. Justify your answer.





(8 marks)

a)

AB, BC, CD, DE, EF, FG, GA are all tracks in the graph

Every component appears exactly once before returning to A

It is a Hamiltonian cycle

b)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F G A

Arcs not in the cycle: AD, BF, CE, CG, EG, AF

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AD: I

BF crosses AD (I), so BF: O

CE crosses AD (I), so CE: O

CG crosses AD (I), so CG: O

CG also crosses BF, which is outside

CG is outside and so is BF, and they cross

CG can be placed neither inside nor outside without a crossing

Not planar: the tracks cannot all go on one side

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c)

AO was labelled first and forced the other labels, so try removing AO and run the algorithm again

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C O E F G A

Arcs not in the cycle: BF, CE, CG, EG, AF

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

BF: I

CG crosses BF (I), so CG: O

EG crosses BF (I), so EG: O

AF crosses CG (O), so AF: I

No more arcs cross those labelled, so choose the next unlabelled arc

CE: I

Inside: BF, AF, CE

Outside: CG, EG

No two inside arcs cross and no two outside arcs cross

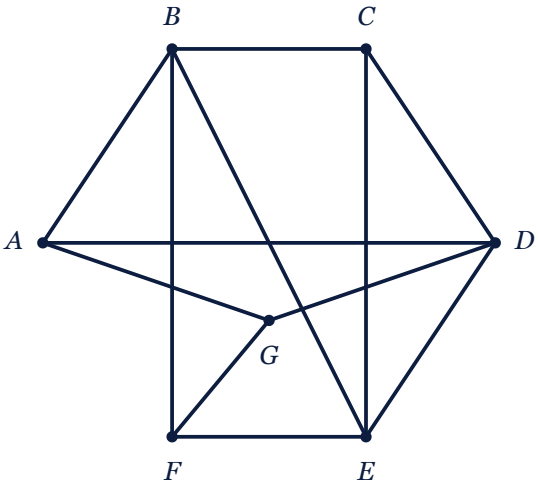
Move track AO: the other twelve can be printed without crossing

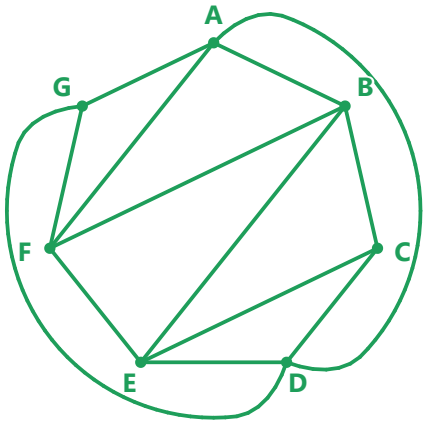
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11 The graph below has seven vertices, A to G, and twelve arcs. A B C D E F G A is a Hamiltonian cycle for the graph.

a. Use the planarity algorithm to show that the graph is planar.

b. One more arc is to be added to the graph. It will be one of AE, AF or CF. Determine which of these arcs could be added so that the graph is still planar. You must justify your answer.





(7 marks)

a)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F G A

Arcs not in the cycle: BE, BF, CE, AD, DG

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

BE: I

AD crosses BE (I), so AD: O

DG crosses BE (I), so DG: O

BF crosses AD (O), so BF: I

CE crosses AD (O), so CE: I

Inside: BE, BF, CE

Outside: AD, DG

No two inside arcs cross and no two outside arcs cross

The graph is planar (drawn without crossings)

b)

An added arc must not cross an inside arc if it goes inside, nor an outside arc if it goes outside

AE crosses: DG (O), BF (I)

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AE crosses an inside arc and an outside arc: no place for it

AF crosses: OG (O)

AF crosses only an outside arc, so it can go inside: the graph stays planar

CF crosses: BE (I), AO (O), OG (O)

CF crosses an inside arc and an outside arc: no place for it

Check by rerunning the algorithm with each arc

With AE: not planar; with AF: planar; with CF: not planar

Only AF can be added

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12 The graph G below has six vertices, A to F, and ten arcs.

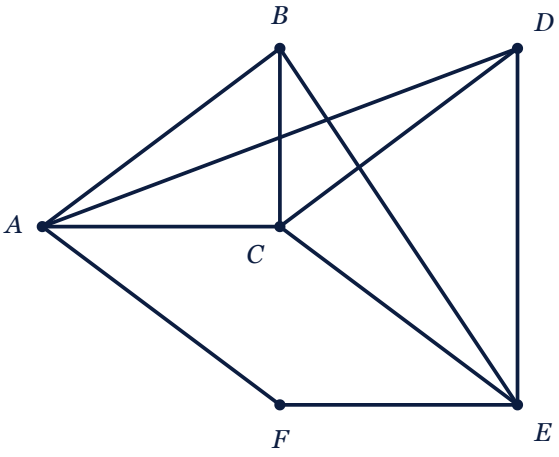
a. State whether G is Eulerian, semi-Eulerian or neither, giving a reason for your answer.

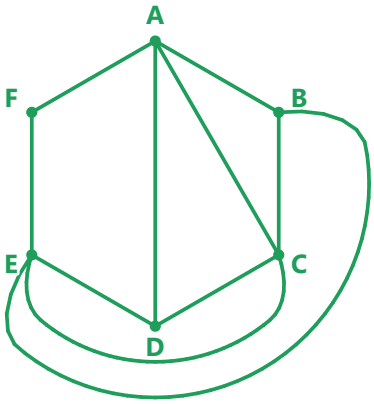
b. Find the number of arcs that would need to be added to G to make the complete graph K_6.

c. Write down a Hamiltonian cycle for G, starting at A.

d. Use your Hamiltonian cycle and the planarity algorithm to show that G is planar.

e. The arc BD is added to G. State, with a reason, whether the new graph is Eulerian, and determine whether it is planar.





(10 marks)

a)

Degrees: A 4, B 3, C 4, D 3, E 4, F 2

Exactly two vertices, B and D, have odd degree

Semi-Eulerian

b)

K6 arcs = 6 × 5 ÷ 2 = 15

15 − 10 = 5

5 arcs

c)

AB, BC, CD, DE, EF, FA are all arcs of G

A B C D E F A

d)

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AD, BE, CE, AC

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AD: I

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BE crosses AD (I), so BE: O

CE crosses AD (I), so CE: O

AC crosses BE (O), so AC: I

Inside: AD, AC

Outside: BE, CE

No two inside arcs cross and no two outside arcs cross

G is planar

e)

B and D now have degree 4 and 4; all degrees are even

Eulerian

Rerun the planarity algorithm with BD included

Draw the Hamiltonian cycle as a polygon

Hamiltonian cycle: A B C D E F A

Arcs not in the cycle: AD, BE, CE, AC, BD

Put the first unlabelled arc inside (I); an arc crossing an inside arc goes outside (O) and an arc crossing an outside arc goes inside

AD: I

BE crosses AD (I), so BE: O

CE crosses AD (I), so CE: O

AC crosses BE (O), so AC: I

BD crosses CE (O), so BD: I

BD also crosses AC, which is inside

BD is inside and so is AC, and they cross

BD can be placed neither inside nor outside without a crossing

With BD added the graph is not planar

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