The simplex algorithm

Worked solutions

Further Maths topic practice

This topic online

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1 A linear programming problem is given as

Maximise $P = 4x + 5y + 3z$

subject to

$3x + 2y + z \leq 60$

$x + 4y + 2z \leq 48$

$2x + y + 3z \leq 45$

$x, y, z \geq 0$

The problem is to be solved using the Simplex algorithm.

a. Using slack variables s_1, s_2 and s_3, write the three constraints as equations.

b. Write the objective function in the form used for the objective row of a Simplex tableau.

c. State the values of x, y, z, s_1, s_2, s_3 and P at the start of the algorithm, and the vertex of the feasible region this represents.

(4 marks)

a)

Add one slack variable to each ≤ constraint

$3x + 2y + z + s1 = 60$

$x + 4y + 2z + s2 = 48$

$2x + y + 3z + s3 = 45$

with $s1, s2, s3 \geqslant 0$

$3x + 2y + z + s1 = 60,\ x + 4y + 2z + s2 = 48,\ 2x + y + 3z + s3 = 45$

b)

Move every term to the left-hand side

$P = 4x + 5y + 3z$

$P - 4x - 5y - 3z = 0$

$P - 4x - 5y - 3z = 0$

c)

Decision variables are non-basic at the start, so they are zero

$x = y = z = 0$

Each slack is then equal to the right-hand side

$s1 = 60,\ s2 = 48,\ s3 = 45$

$P = 0$

$x = y = z = 0,\ s1 = 60,\ s2 = 48,\ s3 = 45,\ P = 0$: the origin $(0, 0, 0)$

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2 A linear programming problem is given as

Maximise $P = 6x + 5y$

subject to

$x + y \leq 12$

$3x + 2y \leq 30$

$x + 3y \leq 27$

$x, y \geq 0$

a. Using slack variables s_1, s_2 and s_3, set up the initial tableau for solving this problem using the Simplex algorithm. Use the template below.

b. Identify the pivot for the first iteration. You must give reasons for your choice of column and row, showing your θ values.

Initial tableau

b.v.	x	y	s_1	s_2	s_3	Value
s_1	1	1	1	0	0	12
s_2	3	2	0	1	0	30
s_3	1	3	0	0	1	27
P	−6	−5	0	0	0	0

(5 marks)

a)

Slack variables: $x + y + s1 = 12$, $3x + 2y + s2 = 30$, $x + 3y + s3 = 27$

Objective row: $P - 6x - 5y = 0$

Initial tableau (columns: x, y, s1, s2, s3 | Value)

s1: 1, 1, 1, 0, 0 | 12

s2: 3, 2, 0, 1, 0 | 30

s3: 1, 3, 0, 0, 1 | 27

P: -6, -5, 0, 0, 0 | 0

Initial tableau as shown

b)

Most negative entry in the P row is -6, in the x column: x enters

θ for R1: $12 \div 1 = 12$

θ for R2: $30 \div 3 = 10$

θ for R3: $27 \div 1 = 27$

Smallest θ is 10 in R2: pivot is 3, x replaces s2

Pivot is the 3 in the s2 row (x column)

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3 The initial tableau for a linear programming problem in x, y and z is shown below. The objective is to maximise P.

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	1	2	2	1	0	0	40
s_2	3	1	2	0	1	0	45
s_3	2	2	1	0	0	1	50
P	−4	−3	−5	0	0	0	0

a. Explain why the z column is chosen as the pivot column for the first iteration.

b. Perform one iteration of the Simplex algorithm. You should show your θ values and state the row operations you use.

c. Explain whether the tableau you obtained in part (b) is optimal.

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
z	$\frac{1}{2}$	1	1	$\frac{1}{2}$	0	0	20	R1 ÷ 2
s_2	2	−1	0	−1	1	0	5	R2 − 2R1
s_3	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	0	1	30	R3 − R1
P	$-\frac{3}{2}$	2	0	$\frac{5}{2}$	0	0	100	R4 + 5R1

(6 marks)

a)

The most negative entry in the P row is -5, in the z column

Increasing z increases P fastest (by 5 per unit)

z: it has the most negative entry, -5, in the objective row

b)

θ for R1: $40 \div 2 = 20$

θ for R2: $45 \div 2 = \frac{45}{2}$

θ for R3: $50 \div 1 = 50$

Smallest θ is 20 in R1: pivot is 2, z replaces s1

Divide the pivot row by the pivot, then clear the rest of the z column using the new R1

Tableau after 1st iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

z: $\frac{1}{2}$, 1, 1, $\frac{1}{2}$, 0, 0 | 20 | R1 ÷ 2

s2: 2, −1, 0, −1, 1, 0 | 5 | R2 − 2R1

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$s3: \frac{3}{2}, 1, 0, -\frac{1}{2}, 0, 1 \mid 30 \mid R3 - R1$

$P: -\frac{3}{2}, 2, 0, \frac{5}{2}, 0, 0 \mid 100 \mid R4 + 5R1$

Tableau after one iteration as shown; $z = 20$, $s2 = 5$, $s3 = 30$, $P = 100$

c)

The P row still has a negative entry: $-\frac{3}{2}$ in the x column

Not optimal: there is a negative value in the objective row

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4 A linear programming problem is given as

Maximise $P = 5x + 4y$

subject to

$x + y \leq 10$

$3x + y \leq 24$

$x + 2y \leq 16$

$x, y \geq 0$

a. Using slack variables, set up the initial tableau for solving this problem with the Simplex algorithm.

b. Solve the problem using the Simplex algorithm. Take the most negative number in the profit row to indicate the pivot column each time, and state the row operations you use.

c. Explain how you know your final tableau is optimal, and state the optimal values of x, y, P and the slack variables.

Use the templates below.

Initial tableau

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
s_1	1	1	1	0	0	10	
s_2	3	1	0	1	0	24	
s_3	1	2	0	0	1	16	
P	−5	−4	0	0	0	0	

Tableau after 1st iteration

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
s_1	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	2	R1 − R2
x	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	8	R2 ÷ 3
s_3	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	1	8	R3 − R2
P	0	$-\frac{7}{3}$	0	$\frac{5}{3}$	0	40	R4 + 5R2

Tableau after 2nd iteration

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
y	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	3	R1 × 3/2
x	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	7	R2 − 1/3 R1
s_3	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	3	R3 − 5/3 R1
P	0	0	$\frac{7}{2}$	$\frac{1}{2}$	0	47	R4 + 7/3 R1

(10 marks)

a)

$x + y + s1 = 10,\ 3x + y + s2 = 24,\ x + 2y + s3 = 16,\ P - 5x - 4y = 0$

Initial tableau (columns: x, y, s1, s2, s3 | Value)

s1: 1, 1, 1, 0, 0 | 10

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s2: 3, 1, 0, 1, 0 | 24

s3: 1, 2, 0, 0, 1 | 16

P: -5, -4, 0, 0, 0 | 0

b)

Most negative entry in the P row is -5, in the x column: x enters

θ for R1: $10 \div 1 = 10$

θ for R2: $24 \div 3 = 8$

θ for R3: $16 \div 1 = 16$

Smallest θ is 8 in R2: pivot is 3, x replaces s2

Divide the pivot row by the pivot, then clear the rest of the x column using the new R2

Tableau after 1st iteration (columns: x, y, s1, s2, s3 | Value | Row ops)

s1: $0, \frac{2}{3}, 1, -\frac{1}{3}, 0 \mid 2 \mid R1 - R2$

x: $1, \frac{1}{3}, 0, \frac{1}{3}, 0 \mid 8 \mid R2 \div 3$

s3: $0, \frac{5}{3}, 0, -\frac{1}{3}, 1 \mid 8 \mid R3 - R2$

P: $0, -\frac{7}{3}, 0, \frac{5}{3}, 0 \mid 40 \mid R4 + 5R2$

Most negative entry in the P row is $-\frac{7}{3}$, in the y column: y enters

θ for R1: $2 \div \frac{2}{3} = 3$

θ for R2: $8 \div \frac{1}{3} = 24$

θ for R3: $8 \div \frac{5}{3} = \frac{24}{5}$

Smallest θ is 3 in R1: pivot is $\frac{2}{3}$, y replaces s1

Tableau after 2nd iteration (columns: x, y, s1, s2, s3 | Value | Row ops)

y: $0, 1, \frac{3}{2}, -\frac{1}{2}, 0 \mid 3 \mid R1 \times \frac{3}{2}$

x: $1, 0, -\frac{1}{2}, \frac{1}{2}, 0 \mid 7 \mid R2 - \frac{1}{3}R1$

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s3: $0, 0, -\frac{5}{2}, \frac{1}{2}, 1 \mid 3 \mid R3 - \frac{5}{3}R1$

P: $0, 0, \frac{7}{2}, \frac{1}{2}, 0 \mid 47 \mid R4 + \frac{7}{3}R1$

c)

No negative entries in the P row, so this tableau is optimal

P row: $P + \frac{7}{2}s1 + \frac{1}{2}s2 = 47$

Increasing s1 or s2 from zero would decrease P

Basic: $x = 7, y = 3, s3 = 3$

Non-basic: $s1 = 0, s2 = 0$

$$\boxed{x = 7, y = 3, P = 47 \ (s1 = 0, s2 = 0, s3 = 3)}$$

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5 A bakery makes loaves of three types: wholemeal, rye and spelt, in batches. It makes x batches of wholemeal, y batches of rye and z batches of spelt each day. The profit, P, in pounds, is to be maximised, where $P = 5x + 4y + 3z$.

The constraints are

mixing time (minutes): $2x + y + 2z \leq 95$
oven time (minutes): $2x + 2y + z \leq 110$
shaping time (minutes): $x + 3y + 3z \leq 115$

where s_1, s_2 and s_3 are the slack variables for these constraints. After applying the Simplex algorithm the tableau below is obtained.

b.v.	x	y	z	s_1	s_2	s_3	Value
x	1	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	0	40
y	0	1	−1	−1	1	0	15
s_3	0	0	$\frac{9}{2}$	2	$-\frac{5}{2}$	1	30
P	0	0	$\frac{1}{2}$	1	$\frac{3}{2}$	0	260

a. Explain how you know that this tableau is optimal.

b. State the number of batches of each type the bakery should make and the maximum daily profit.

c. State the values of the slack variables and explain what they mean in context.

d. Write down the equation represented by the profit row and use it to explain why making any batches of spelt would reduce the profit.

(7 marks)

a)

Every entry in the P row is ≥ 0

No negative values in the objective row, so optimal

b)

Read the basic variables from the Value column; non-basic variables are zero

$x = 40$

$y = 15$

$z = 0$ (non-basic)

$P = 260$

40 batches of wholemeal, 15 of rye, 0 of spelt; profit £260

c)

$s1 = 0$, $s2 = 0$

All the mixing time and all the oven time are used

$s3 = 30$

30 minutes of shaping time are not used each day

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d)

$$P + \frac{1}{2}z + s1 + \frac{3}{2}s2 = 260$$

$$P = 260 - \frac{1}{2}z - s1 - \frac{3}{2}s2$$

z, $s1$ and $s2$ are all $\geqslant 0$

Making $z = 1$ would give $P = 260 - \frac{1}{2} = 259.5$

Each batch of spelt lowers the profit by £0.50

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6 The initial tableau for a linear programming problem in x, y and z is shown below. The objective is to maximise P.

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	2	3	1	1	0	0	36
s_2	1	1	2	0	1	0	20
s_3	3	2	4	0	0	1	48
P	−3	−5	−4	0	0	0	0

a. Solve the problem using the Simplex algorithm, taking the most negative number in the profit row to indicate the pivot column at each stage. You must show your θ values and state the row operations you use. Use the templates below.

b. Explain how you know your final tableau is optimal and state the optimal values of x, y, z and P.

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
y	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	12	R1 ÷ 3
s_2	$\frac{1}{3}$	0	$\frac{5}{3}$	$-\frac{1}{3}$	1	0	8	R2 − R1
s_3	$\frac{5}{3}$	0	$\frac{10}{3}$	$-\frac{2}{3}$	0	1	24	R3 − 2R1
P	$\frac{1}{3}$	0	$-\frac{7}{3}$	$\frac{5}{3}$	0	0	60	R4 + 5R1

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
y	$\frac{3}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{52}{5}$	R1 − 1/3 R2
z	$\frac{1}{5}$	0	1	$-\frac{1}{5}$	$\frac{3}{5}$	0	$\frac{24}{5}$	R2 × 3/5
s_3	1	0	0	0	−2	1	8	R3 − 10/3 R2
P	$\frac{4}{5}$	0	0	$\frac{6}{5}$	$\frac{7}{5}$	0	$\frac{356}{5}$	R4 + 7/3 R2

(8 marks)

a)

Most negative entry in the P row is −5, in the y column: y enters

θ for R1: 36 ÷ 3 = 12

θ for R2: 20 ÷ 1 = 20

θ for R3: 48 ÷ 2 = 24

Smallest θ is 12 in R1: pivot is 3, y replaces s1

Divide the pivot row by the pivot, then clear the rest of the y column using the new R1

Tableau after 1st iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

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$y: \frac{2}{3}, 1, \frac{1}{3}, \frac{1}{3}, 0, 0 \mid 12 \mid R1 \div 3$

$s2: \frac{1}{3}, 0, \frac{5}{3}, -\frac{1}{3}, 1, 0 \mid 8 \mid R2 - R1$

$s3: \frac{5}{3}, 0, \frac{10}{3}, -\frac{2}{3}, 0, 1 \mid 24 \mid R3 - 2R1$

$P: \frac{1}{3}, 0, -\frac{7}{3}, \frac{5}{3}, 0, 0 \mid 60 \mid R4 + 5R1$

Most negative entry in the P row is $-\frac{7}{3}$, in the z column: z enters

θ for R1: $12 \div \frac{1}{3} = 36$

θ for R2: $8 \div \frac{5}{3} = \frac{24}{5}$

θ for R3: $24 \div \frac{10}{3} = \frac{36}{5}$

Smallest θ is $\frac{24}{5}$ in R2: pivot is $\frac{5}{3}$, z replaces s2

Tableau after 2nd iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

$y: \frac{3}{5}, 1, 0, \frac{2}{5}, -\frac{1}{5}, 0 \mid \frac{52}{5} \mid R1 - \frac{1}{3}R2$

$z: \frac{1}{5}, 0, 1, -\frac{1}{5}, \frac{3}{5}, 0 \mid \frac{24}{5} \mid R2 \times \frac{3}{5}$

$s3: 1, 0, 0, 0, -2, 1 \mid 8 \mid R3 - \frac{10}{3}R2$

$P: \frac{4}{5}, 0, 0, \frac{6}{5}, \frac{7}{5}, 0 \mid \frac{356}{5} \mid R4 + \frac{7}{3}R2$

b)

No negative values in the P row, so optimal

Basic: $y = \frac{52}{5}, z = \frac{24}{5}, s3 = 8$

Non-basic: $x = 0, s1 = 0, s2 = 0$

$$\boxed{x = 0, y = \frac{52}{5}, z = \frac{24}{5}, P = \frac{356}{5}}$$

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7 A maximisation linear programming problem in x, y and z is being solved using the Simplex algorithm. The slack variables are s_1, s_2 and s_3. The tableau after the first iteration is shown below.

b.v.	x	y	z	s_1	s_2	s_3	Value
s_1	1	0	3	1	−1	0	20
y	$\frac{1}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	20
s_3	$\frac{3}{4}$	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	1	75
P	−4	0	0	0	2	0	160

a. State the column that contained the pivot for the first iteration. You must give a reason.

b. By considering the row operations that were used, formulate the original linear programming problem. State the objective and list the constraints as inequalities with integer coefficients.

c. Perform the second iteration, showing your θ values and the row operations you use.

d. State whether the tableau you obtained in part (c) is optimal, giving a reason.

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
x	1	0	3	1	−1	0	20	R1 ÷ 1
y	0	1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	15	R2 − 1/4 R1
s_3	0	0	$-\frac{3}{2}$	$-\frac{3}{4}$	$\frac{1}{2}$	1	60	R3 − 3/4 R1
P	0	0	12	4	−2	0	240	R4 + 4R1

(10 marks)

a)

y is basic in the tableau, replacing s2

The y column: y is now a basic variable (in the s2 row)

b)

The y row is the old R2 divided by the pivot

y row: y: $\frac{1}{4}$, 1, $\frac{1}{4}$, 0, $\frac{1}{4}$, 0 | 20

Its s2 entry is $\frac{1}{4}$, so the pivot was 4: multiply by 4

$x + 4y + z + s2 = 80$

s2 was 0 in every other row, so each s2 entry now shows the multiple of the y row that was subtracted

s1 row: s1: 1, 0, 3, 1, −1, 0 | 20

Its s2 entry is $-1 = -4 \times \frac{1}{4}$, so 4 x y row was subtracted

Old R1 = new R1 + 4 x (y row)

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$2x + 4y + 4z + s1 = 100$

s3 row: s3: $\frac{3}{4}, 0, \frac{3}{4}, 0, -\frac{1}{4}, 1 \mid 75$

s2 entry $-\frac{1}{4}$, so 1 x y row was subtracted

Old R3 = new R3 + 1 x (y row)

$x + y + z + s3 = 95$

P row: P: $-4, 0, 0, 0, 2, 0 \mid 160$

s2 entry $2 = 8 \times \frac{1}{4}$, so 8 x y row was added

Old P row = new P row - 8 x (y row)

$P - 6x - 8y - 2z = 0$

Maximise $P = 6x + 8y + 2z$ subject to $2x + 4y + 4z \leq 100$, $x + 4y + z \leq 80$, $x + y + z \leq 95$, $x, y, z \geq 0$

c)

Most negative entry in the P row is -4, in the x column: x enters

θ for R1: $20 \div 1 = 20$

θ for R2: $20 \div \frac{1}{4} = 80$

θ for R3: $75 \div \frac{3}{4} = 100$

Smallest θ is 20 in R1: pivot is 1, x replaces s1

Tableau after 2nd iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

x: $1, 0, 3, 1, -1, 0 \mid 20 \mid R1 \div 1$

y: $0, 1, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}, 0 \mid 15 \mid R2 - \frac{1}{4}R1$

s3: $0, 0, -\frac{3}{2}, -\frac{3}{4}, \frac{1}{2}, 1 \mid 60 \mid R3 - \frac{3}{4}R1$

P: $0, 0, 12, 4, -2, 0 \mid 240 \mid R4 + 4R1$

$x = 20, y = 15, s3 = 60, P = 240$

d)

P row still has -2 in the s2 column

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Not optimal: there is still a negative value in the objective row

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8 A linear programming problem is given as

Maximise $P = x + ky$

subject to

$x + 2y \leq 16$

$3x + y \leq 18$

$x - y \leq 4$

$x, y \geq 0$

where k is a positive constant. After two iterations of the Simplex algorithm a possible tableau is

b.v.	x	y	s_1	s_2	s_3	Value
y	0	1	$\frac{3}{5}$	$-\frac{1}{5}$	0	6
x	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	4
s_3	0	0	$\frac{4}{5}$	$-\frac{3}{5}$	1	6
P	0	0	$\frac{-1+3k}{5}$	$\frac{2-k}{5}$	0	$4 + 6k$

a. State the values of x, y and P given by this tableau, in terms of k where necessary.

b. Find the range of values of k for which this tableau is optimal.

c. Given that $k = 3$, identify the pivot for the next iteration. You must give reasons for your answer. (8 marks)

a)

Read the basic variables from the Value column

$x = 4$

$y = 6$

$P = 4 + 6k$

Check: $P = x + ky = 4 + 6k$

$\boxed{x = 4,\ y = 6,\ P = 4 + 6k}$

b)

Optimal when no entry in the P row is negative

s1 column: $\frac{-1 + 3k}{5} \geqslant 0$

$3k - 1 \geqslant 0$

$k \geqslant \frac{1}{3}$

s2 column: $\frac{2 - k}{5} \geqslant 0$

$2 - k \geqslant 0$

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$k \leq 2$

$$\boxed{\frac{1}{3} \leq k \leq 2}$$

c)

Substitute $k = 3$ into the P row

s1 entry $= \frac{3 \times 3 - 1}{5} = \frac{8}{5}$

s2 entry $= \frac{2 - 3}{5} = -\frac{1}{5}$

$-\frac{1}{5}$ is the only negative entry, so s2 enters

R1: s2 entry $-\frac{1}{5}$ is not positive, no θ

θ for R2: $4 \div \frac{2}{5} = 10$

R3: s2 entry $-\frac{3}{5}$ is not positive, no θ

$$\boxed{\text{Pivot is the } \frac{2}{5} \text{ in the s2 column, x row}}$$

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9 A linear programming problem is given as

Minimise $C = 2x - 3y - 4z$

subject to

$2x + y + z \leq 30$

$3x + y + 2z \leq 75$

$2x + 2y + 3z \leq 80$

$x, y, z \geq 0$

a. Explain how this problem can be rewritten as a maximisation problem, stating the new objective.

b. Set up the initial tableau for solving the problem with the Simplex algorithm, using slack variables s_1, s_2 and s_3.

c. Solve the problem, taking the most negative number in the objective row to indicate the pivot column each time. State the row operations you use and the optimal values of x, y, z and C.

Use the templates below.

Initial tableau

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
s_1	2	1	1	1	0	0	30	
s_2	3	1	2	0	1	0	75	
s_3	2	2	3	0	0	1	80	
P	2	−3	−4	0	0	0	0	

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
s_1	$\frac{4}{3}$	$\frac{1}{3}$	0	1	0	$-\frac{1}{3}$	$\frac{10}{3}$	R1 − R3
s_2	$\frac{5}{3}$	$-\frac{1}{3}$	0	0	1	$-\frac{2}{3}$	$\frac{65}{3}$	R2 − 2R3
z	$\frac{2}{3}$	$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	$\frac{80}{3}$	R3 ÷ 3
P	$\frac{14}{3}$	$-\frac{1}{3}$	0	0	0	$\frac{4}{3}$	$\frac{320}{3}$	R4 + 4R3

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
y	4	1	0	3	0	−1	10	R1 × 3
s_2	3	0	0	1	1	−1	25	R2 + 1/3 R1
z	−2	0	1	−2	0	1	20	R3 − 2/3 R1
P	6	0	0	1	0	1	110	R4 + 1/3 R1

(9 marks)

a)

Minimising C is the same as maximising $P = -C$

$$P = -C = -2x + 3y + 4z$$

$$P + 2x - 3y - 4z = 0$$

Maximise $P = -2x + 3y + 4z$

b)

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Initial tableau (columns: x, y, z, s1, s2, s3 | Value)

s1: 2, 1, 1, 1, 0, 0 | 30

s2: 3, 1, 2, 0, 1, 0 | 75

s3: 2, 2, 3, 0, 0, 1 | 80

P: 2, -3, -4, 0, 0, 0 | 0

c)

Most negative entry in the P row is -4, in the z column: z enters

θ for R1: $30 \div 1 = 30$

θ for R2: $75 \div 2 = \frac{75}{2}$

θ for R3: $80 \div 3 = \frac{80}{3}$

Smallest θ is $\frac{80}{3}$ in R3: pivot is 3, z replaces s3

Divide the pivot row by the pivot, then clear the rest of the z column using the new R3

Tableau after 1st iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

s1: $\frac{4}{3}, \frac{1}{3}, 0, 1, 0, -\frac{1}{3}$ | $\frac{10}{3}$ | R1 - R3

s2: $\frac{5}{3}, -\frac{1}{3}, 0, 0, 1, -\frac{2}{3}$ | $\frac{65}{3}$ | R2 - 2R3

z: $\frac{2}{3}, \frac{2}{3}, 1, 0, 0, \frac{1}{3}$ | $\frac{80}{3}$ | R3 ÷ 3

P: $\frac{14}{3}, -\frac{1}{3}, 0, 0, 0, \frac{4}{3}$ | $\frac{320}{3}$ | R4 + 4R3

Most negative entry in the P row is $-\frac{1}{3}$, in the y column: y enters

θ for R1: $\frac{10}{3} \div \frac{1}{3} = 10$

θ for R3: $\frac{80}{3} \div \frac{2}{3} = 40$

R2: entry not positive, no θ

Smallest θ is 10 in R1: pivot is $\frac{1}{3}$, y replaces s1

Tableau after 2nd iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

y: 4, 1, 0, 3, 0, -1 | 10 | R1 × 3

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$s2: 3, 0, 0, 1, 1, -1 \mid 25 \mid R2 + \frac{1}{3}R1$

$z: -2, 0, 1, -2, 0, 1 \mid 20 \mid R3 - \frac{2}{3}R1$

$P: 6, 0, 0, 1, 0, 1 \mid 110 \mid R4 + \frac{1}{3}R1$

No negative values in the P row, so optimal

Basic: $y = 10$, $s2 = 25$, $z = 20$

Non-basic: $x = 0$, $s1 = 0$, $s3 = 0$

$P = 110$

$C = -P = -110$

$x = 0$, $y = 10$, $z = 20$; minimum $C = -110$

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10 A workshop makes garden chairs, stools and benches. Each item passes through three processes. The time, in hours, each item needs for each process and the time available each week are shown in the table.

	Chair	Stool	Bench	Hours available
Cutting	2	1	1	65
Sanding	2	3	1	80
Finishing	3	2	2	110

The profit is £9 on each chair, £5 on each stool and £6 on each bench. The workshop wants to maximise its weekly profit, £P, by making x chairs, y stools and z benches.

a. Formulate this as a linear programming problem.

b. Using slack variables s_1, s_2 and s_3, set up the initial tableau and solve the problem using the Simplex algorithm, taking the most negative number in the profit row to indicate the pivot column each time. State the row operations you use. Use the templates below.

c. Interpret your final tableau in context, including the value of any slack variable that is not zero.

Initial tableau

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
s_1	2	1	1	1	0	0	65	
s_2	2	3	1	0	1	0	80	
s_3	3	2	2	0	0	1	110	
P	−9	−5	−6	0	0	0	0	

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
x	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{65}{2}$	R1 ÷ 2
s_2	0	2	0	−1	1	0	15	R2 − 2R1
s_3	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	1	$\frac{25}{2}$	R3 − 3R1
P	0	$-\frac{1}{2}$	$-\frac{3}{2}$	$\frac{9}{2}$	0	0	$\frac{585}{2}$	R4 + 9R1

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
x	1	0	0	2	0	−1	20	R1 − 1/2 R3
s_2	0	2	0	−1	1	0	15	R2
z	0	1	1	−3	0	2	25	R3 × 2
P	0	1	0	0	0	3	330	R4 + 3/2 R3

(12 marks)

a)

Profit: £9 per chair, £5 per stool, £6 per bench

$P = 9x + 5y + 6z$

One constraint per process, in hours

Cutting: $2x + y + z \leq 65$

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Sanding: $2x + 3y + z \leq 80$

Finishing: $3x + 2y + 2z \leq 110$

$x, y, z \geq 0$

Maximise $P = 9x + 5y + 6z$ subject to $2x + y + z \leq 65$, $2x + 3y + z \leq 80$, $3x + 2y + 2z \leq 110$, $x, y, z \geq 0$

b)

Initial tableau (columns: x, y, z, s1, s2, s3 | Value)

s1: 2, 1, 1, 1, 0, 0 | 65

s2: 2, 3, 1, 0, 1, 0 | 80

s3: 3, 2, 2, 0, 0, 1 | 110

P: −9, −5, −6, 0, 0, 0 | 0

Most negative entry in the P row is −9, in the x column: x enters

θ for R1: $65 \div 2 = \frac{65}{2}$

θ for R2: $80 \div 2 = 40$

θ for R3: $110 \div 3 = \frac{110}{3}$

Smallest θ is $\frac{65}{2}$ in R1: pivot is 2, x replaces s1

Divide the pivot row by the pivot, then clear the rest of the x column using the new R1

Tableau after 1st iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

x: $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0 \mid \frac{65}{2} \mid R1 \div 2$

s2: $0, 2, 0, -1, 1, 0 \mid 15 \mid R2 - 2R1$

s3: $0, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, 0, 1 \mid \frac{25}{2} \mid R3 - 3R1$

P: $0, -\frac{1}{2}, -\frac{3}{2}, \frac{9}{2}, 0, 0 \mid \frac{585}{2} \mid R4 + 9R1$

Most negative entry in the P row is $-\frac{3}{2}$, in the z column: z enters

θ for R1: $\frac{65}{2} \div \frac{1}{2} = 65$

θ for R3: $\frac{25}{2} \div \frac{1}{2} = 25$

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R2: entry not positive, no θ

Smallest θ is 25 in R3: pivot is $\frac{1}{2}$, z replaces s3

Tableau after 2nd iteration (columns: x, y, z, s1, s2, s3 | Value | Row ops)

x: 1, 0, 0, 2, 0, -1 | 20 | R1 - $\frac{1}{2}$R3

s2: 0, 2, 0, -1, 1, 0 | 15 | R2

z: 0, 1, 1, -3, 0, 2 | 25 | R3 × 2

P: 0, 1, 0, 0, 0, 3 | 330 | R4 + $\frac{3}{2}$R3

No negative values in the P row, so optimal

Basic: $x = 20$, $s2 = 15$, $z = 25$

Non-basic: $y = 0$, $s1 = 0$, $s3 = 0$

$x = 20$, $y = 0$, $z = 25$, $P = 330$

c)

Make 20 chairs, no stools and 25 benches: profit £330

$s1 = 0$, $s3 = 0$: all cutting and finishing time used

s2 = 15: 15 hours of sanding time are not used

(The 0 under non-basic s1 in the P row shows another optimum exists, e.g. 55 benches only, also £330)

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11 Consider the linear programming problem

Maximise $P = 3x + 2y + 4z$

subject to

$2x + y + z \leq 53$

$x + 3y + z \leq 37$

$x + 2y + 2z \leq 50$

$x \geq 2, y \geq 3, z \geq 1$

a. Explain why the Simplex algorithm cannot be used to solve this problem in its current form.

b. Use the substitutions $x = X + 2$, $y = Y + 3$, $z = Z + 1$ and $P = Q + 16$ to reformulate the problem in X, Y and Z so that the Simplex algorithm can be used. Write the constraints as simplified inequalities with integer coefficients.

c. Using slack variables s_1, s_2 and s_3, solve the reformulated problem with the Simplex algorithm, taking the most negative number in the objective row each time. State the row operations you use. Use the templates below.

d. Hence state the optimal values of x, y, z and P.

Initial tableau

b.v.	X	Y	Z	s_1	s_2	s_3	Value	Row ops
s_1	2	1	1	1	0	0	45	
s_2	1	3	1	0	1	0	25	
s_3	1	2	2	0	0	1	40	
Q	−3	−2	−4	0	0	0	0	

Tableau after 1st iteration

b.v.	X	Y	Z	s_1	s_2	s_3	Value	Row ops
s_1	$\frac{3}{2}$	0	0	1	0	$-\frac{1}{2}$	25	R1 − R3
s_2	$\frac{1}{2}$	2	0	0	1	$-\frac{1}{2}$	5	R2 − R3
Z	$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	20	R3 ÷ 2
Q	−1	2	0	0	0	2	80	R4 + 4R3

Tableau after 2nd iteration

b.v.	X	Y	Z	s_1	s_2	s_3	Value	Row ops
s_1	0	−6	0	1	−3	1	10	R1 − 3/2 R2
X	1	4	0	0	2	−1	10	R2 × 2
Z	0	−1	1	0	−1	1	15	R3 − 1/2 R2
Q	0	6	0	0	2	1	90	R4 + R2

(12 marks)

a)

The constraints $x \geq 2$, $y \geq 3$, $z \geq 1$ are not of the form $\leq$

The origin is not feasible, so the algorithm has no starting vertex

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b)

Substitute $x = X + 2$, $y = Y + 3$, $z = Z + 1$

$2(X + 2) + (Y + 3) + (Z + 1) \leq 53$

$2X + Y + Z \leq 45$

$(X + 2) + 3(Y + 3) + (Z + 1) \leq 37$

$X + 3Y + Z \leq 25$

$(X + 2) + 2(Y + 3) + 2(Z + 1) \leq 50$

$X + 2Y + 2Z \leq 40$

$P = 3(X + 2) + 2(Y + 3) + 4(Z + 1)$

$P = 3X + 2Y + 4Z + 16$

$Q = P - 16 = 3X + 2Y + 4Z$

Maximise $Q = 3X + 2Y + 4Z$ subject to $2X + Y + Z \leq 45$, $X + 3Y + Z \leq 25$, $X + 2Y + 2Z \leq 40$, $X, Y, Z \geq 0$

c)

Initial tableau (columns: X, Y, Z, s1, s2, s3 | Value)

s1: 2, 1, 1, 1, 0, 0 | 45

s2: 1, 3, 1, 0, 1, 0 | 25

s3: 1, 2, 2, 0, 0, 1 | 40

Q: -3, -2, -4, 0, 0, 0 | 0

Most negative entry in the Q row is -4, in the Z column: Z enters

θ for R1: $45 \div 1 = 45$

θ for R2: $25 \div 1 = 25$

θ for R3: $40 \div 2 = 20$

Smallest θ is 20 in R3: pivot is 2, Z replaces s3

Divide the pivot row by the pivot, then clear the rest of the Z column using the new R3

Tableau after 1st iteration (columns: X, Y, Z, s1, s2, s3 | Value | Row ops)

s1: $\frac{3}{2}$, 0, 0, 1, 0, $-\frac{1}{2}$ | 25 | R1 - R3

s2: $\frac{1}{2}$, 2, 0, 0, 1, $-\frac{1}{2}$ | 5 | R2 - R3

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Z: $\frac{1}{2}, 1, 1, 0, 0, \frac{1}{2} \mid 20 \mid R3 \div 2$

Q: $-1, 2, 0, 0, 0, 2 \mid 80 \mid R4 + 4R3$

Most negative entry in the Q row is -1, in the X column: X enters

θ for R1: $25 \div \frac{3}{2} = \frac{50}{3}$

θ for R2: $5 \div \frac{1}{2} = 10$

θ for R3: $20 \div \frac{1}{2} = 40$

Smallest θ is 10 in R2: pivot is $\frac{1}{2}$, X replaces s2

Tableau after 2nd iteration (columns: X, Y, Z, s1, s2, s3 | Value | Row ops)

s1: $0, -6, 0, 1, -3, 1 \mid 10 \mid R1 - \frac{3}{2}R2$

X: $1, 4, 0, 0, 2, -1 \mid 10 \mid R2 \times 2$

Z: $0, -1, 1, 0, -1, 1 \mid 15 \mid R3 - \frac{1}{2}R2$

Q: $0, 6, 0, 0, 2, 1 \mid 90 \mid R4 + R2$

No negative values in the Q row, so optimal

Basic: $s1 = 10, X = 10, Z = 15$

$Q = 90, Y = 0$

$$\boxed{X = 10, Y = 0, Z = 15, Q = 90}$$

d)

$x = 10 + 2 = 12$

$y = 0 + 3 = 3$

$z = 15 + 1 = 16$

$P = 90 + 16 = 106$

$$\boxed{x = 12, y = 3, z = 16, P = 106}$$

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12 A linear programming problem is given as

Maximise $P = 4x + 6y$

subject to

$2x + 3y \leq 30$

$x + y \leq 12$

$x \leq 8$

$x, y \geq 0$

a. Using slack variables s_1, s_2 and s_3, set up the initial tableau and perform one iteration of the Simplex algorithm, taking the most negative number in the objective row to indicate the pivot column. State the row operations you use.

b. Explain why the tableau you obtained in part (a) is optimal.

c. Explain how the tableau shows that the optimal solution is not unique.

d. Perform a further iteration, with x entering the basis, to find a second optimal vertex.

e. Describe the complete set of optimal solutions.

Use the templates below for parts (a) and (d).

Initial tableau

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
s_1	2	3	1	0	0	30	
s_2	1	1	0	1	0	12	
s_3	1	0	0	0	1	8	
P	−4	−6	0	0	0	0	

Tableau after 1st iteration

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
y	$\frac{2}{3}$	1	$\frac{1}{3}$	0	0	10	R1 ÷ 3
s_2	$\frac{1}{3}$	0	$-\frac{1}{3}$	1	0	2	R2 − R1
s_3	1	0	0	0	1	8	R3
P	0	0	2	0	0	60	R4 + 6R1

Tableau after 2nd iteration

b.v.	x	y	s_1	s_2	s_3	Value	Row ops
y	0	1	1	−2	0	6	R1 − 2/3 R2
x	1	0	−1	3	0	6	R2 × 3
s_3	0	0	1	−3	1	2	R3 − R2
P	0	0	2	0	0	60	R4

(11 marks)

a)

Initial tableau (columns: x, y, s1, s2, s3 | Value)

s1: 2, 3, 1, 0, 0 | 30

s2: 1, 1, 0, 1, 0 | 12

s3: 1, 0, 0, 0, 1 | 8

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P: $-4, -6, 0, 0, 0 \mid 0$

Most negative entry in the P row is -6, in the y column: y enters

θ for R1: $30 \div 3 = 10$

θ for R2: $12 \div 1 = 12$

R3: entry not positive, no θ

Smallest θ is 10 in R1: pivot is 3, y replaces s1

Divide the pivot row by the pivot, then clear the rest of the y column using the new R1

Tableau after 1st iteration (columns: x, y, s1, s2, s3 | Value | Row ops)

y: $\frac{2}{3}, 1, \frac{1}{3}, 0, 0 \mid 10 \mid R1 \div 3$

s2: $\frac{1}{3}, 0, -\frac{1}{3}, 1, 0 \mid 2 \mid R2 - R1$

s3: $1, 0, 0, 0, 1 \mid 8 \mid R3$

P: $0, 0, 2, 0, 0 \mid 60 \mid R4 + 6R1$

$y = 10, s2 = 2, s3 = 8, x = 0, P = 60$

b)

The P row has no negative entries

Optimal: no negative values in the objective row

c)

x is non-basic but its entry in the P row is 0

P row: $P + 2s1 = 60$, so x can increase without changing P

Not unique: a non-basic variable (x) has a zero in the objective row

d)

x is non-basic with 0 in the P row: bring x in

θ for R1: $10 \div \frac{2}{3} = 15$

θ for R2: $2 \div \frac{1}{3} = 6$

θ for R3: $8 \div 1 = 8$

Smallest θ is 6 in R2: pivot is $\frac{1}{3}$, x replaces s2

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Tableau after 2nd iteration (columns: x, y, $s1$, $s2$, $s3$ | Value | Row ops)

y: 0, 1, 1, −2, 0 | 6 | $R1 - \frac{2}{3}R2$

x: 1, 0, −1, 3, 0 | 6 | $R2 \times 3$

$s3$: 0, 0, 1, −3, 1 | 2 | $R3 - R2$

P: 0, 0, 2, 0, 0 | 60 | $R4$

$y = 6,\ x = 6,\ s3 = 2,\ P = 60$

e)

Both vertices (0, 10) and (6, 6) give $P = 60$

Every point on the line $2x + 3y = 30$ between them gives $P = 60$

Optimal at every point on the segment from (0, 10) to (6, 6), $P = 60$

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