Travelling salesman: lower bounds

Worked solutions

Further Maths topic practice

This topic online

1 A travelling salesman problem has seven vertices, A to G. Vertex D and all of its arcs are deleted, and a minimum spanning tree for the remaining six vertices has weight 74

The arcs from D have weights 23, 18, 31, 40, 27 and 35

Find a lower bound for the length of the optimal tour. (2 marks)

Lower bound = RMST weight + two least arcs from the deleted vertex

Arcs from D in order: 18, 23, 27, 31, 35, 40

Two least: 18 and 23

Lower bound = 74 + 18 + 23 = 115

115

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2 The table shows the least distances, in km, between five bus garages, A to E, in Bristol.

	A	B	C	D	E
A	–	29	13	48	18
B	29	–	20	70	47
C	13	20	–	56	31
D	48	70	56	–	55
E	18	47	31	55	–

By deleting E and all of its arcs, find a lower bound for the length of a tour of the five garages.
Complete the table below.

Deleted vertex method

Deleted vertex	RMST weight	Two least arcs	Lower bound
E	81	EA 18, EC 31	130

(3 marks)

Delete E and all of its arcs

Find a minimum spanning tree of the remaining 4 vertices (Prim)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 13 in column A, row C: add AC

Delete row C, number column C as 2

Smallest is 20 in column C, row B: add CB

Delete row B, number column B as 3

Smallest is 48 in column A, row D: add AD

Delete row D, number column D as 4

Weight = 13 + 20 + 48 = 81

Arcs from E: EA 18, EB 47, EC 31, ED 55

Two least: EA 18 and EC 31

Lower bound = 81 + 18 + 31 = 130

Lower bound 130 km

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3 The deleted vertex method is used to find lower bounds for a travelling salesman problem.

a. Explain why the method gives a lower bound for the length of the optimal tour.

Deleting different vertices gives lower bounds of 147, 152 and 139. The nearest neighbour algorithm from two vertices gives upper bounds of 186 and 179

b. Write down the smallest interval that must contain the length, L, of the optimal tour, giving reasons for your choice of each end.

(4 marks)

a)

Any tour visits the deleted vertex once, using two of its arcs

Those two arcs ⩾ the two least arcs from it

Removing the vertex leaves a path through the other vertices

A path through all of them is a spanning tree, so its weight ⩾ RMST weight

So every tour ⩾ RMST + two least arcs

Every tour, so the optimal one, is at least this value

b)

$L \geqslant 147$, $L \geqslant 152$ and $L \geqslant 139$, so $L \geqslant 152$ (Largest lower bound)

$L \leqslant 186$ and $L \leqslant 179$, so $L \leqslant 179$ (smallest upper bound)

$152 \leqslant L \leqslant 179$

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4 The table below shows the least distances, in miles, between six market towns, A to F, in Norfolk.

a. By deleting A and all of its arcs, use Prim's algorithm, starting at B, to find a minimum spanning tree for the remaining five towns. State the order in which you select the arcs and the weight of the tree.

b. Hence find a lower bound for the length of a tour of all six towns.

		1	3	2	5	4
	A	**B**	**C**	**D**	**E**	**F**
A	-	40	52	48	42	68
B	40	-	55	53	57	67
C	52	55	-	11	32	16
D	48	53	11	-	21	25
E	42	57	32	21	-	46
F	68	67	16	25	46	-

Deleted vertex method

Deleted vertex	RMST weight	Two least arcs	Lower bound
A	101	AB 40, AE 42	183

(5 marks)

a)

Delete A and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Start at B: delete row B, number column B as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 53 in column B, row D: add BD

Delete row D, number column D as 2

Smallest is 11 in column D, row C: add DC

Delete row C, number column C as 3

Smallest is 16 in column C, row F: add CF

Delete row F, number column F as 4

Smallest is 21 in column D, row E: add DE

Delete row E, number column E as 5

Weight = 53 + 11 + 16 + 21 = 101

BD, DC, CF, DE; weight 101

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b)

Arcs from A: AB 40, AC 52, AD 48, AE 42, AF 68

Two least: AB 40 and AE 42

Lower bound = 101 + 40 + 42 = 183

183 miles

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5 The table shows the least distances, in km, between six offices, A to F, of a firm in Glasgow.

	A	B	C	D	E	F
A	–	19	58	45	51	31
B	19	–	39	40	32	18
C	58	39	–	42	60	36
D	45	40	42	–	71	22
E	51	32	60	71	–	50
F	31	18	36	22	50	–

a. By deleting A, find a lower bound for the length of a tour of the six offices.

b. By deleting E, find a second lower bound.

c. State the better lower bound, giving a reason.

Deleted vertex method

Deleted vertex	RMST weight	Two least arcs	Lower bound
A	108	AB 19, AF 31	158
E	95	EB 32, EF 50	177

(6 marks)

a)

Delete A and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: BF 18, FD 22, BE 32, FC 36

RMST weight = 18 + 22 + 32 + 36 = 108

Arcs from A: AB 19, AC 58, AD 45, AE 51, AF 31

Two least: AB 19 and AF 31

Lower bound = 108 + 19 + 31 = 158

158 km

b)

Delete E and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: AB 19, BF 18, FD 22, FC 36

RMST weight = 19 + 18 + 22 + 36 = 95

Arcs from E: EA 51, EB 32, EC 60, ED 71, EF 50

Two least: EB 32 and EF 50

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Lower bound = 95 + 32 + 50 = 177

177 km

c)

The optimal tour ⩾ 158 and ⩾ 177

The larger lower bound is closer to the optimal value

177 km

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6 The table shows the least times, in minutes, for a minibus to travel between six schools, A to F, in Swansea.

	A	B	C	D	E	F
A	–	19	30	41	11	46
B	19	–	49	35	25	29
C	30	49	–	65	37	72
D	41	35	65	–	31	51
E	11	25	37	31	–	48
F	46	29	72	51	48	–

a. Use the nearest neighbour algorithm, starting at A, to find an upper bound for the length of a tour of the schools.

b. By deleting F, find a lower bound for the length of the tour.

c. Hence write down an interval within which the length, T minutes, of the optimal tour must lie.

Upper bound

Start	Tour	Length
A	A - E - B - F - D - C - A	211

Lower bound

Deleted vertex	RMST weight	Two least arcs	Lower bound
F	91	FB 29, FA 46	166

(6 marks)

a)

Nearest neighbour from A: each time go to the nearest vertex not yet visited

From A: E 11, B 19, C 30, D 41, F 46, nearest is E

From E: B 25, D 31, C 37, F 48, nearest is B

From B: F 29, D 35, C 49, nearest is F

From F: D 51, C 72, nearest is D

From D: only C is left (65)

ALL vertices visited: return C to A (30)

Tour: A - E - B - F - D - C - A

Length = 11 + 25 + 29 + 51 + 65 + 30 = 211

A - E - B - F - D - C - A, 211 minutes

b)

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Delete F and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: AE 11, AB 19, AC 30, ED 31

RMST weight = 11 + 19 + 30 + 31 = 91

Arcs from F: FA 46, FB 29, FC 72, FD 51, FE 48

Two least: FB 29 and FA 46

Lower bound = 91 + 29 + 46 = 166

166 minutes

c)

Lower bound ≤ optimal ≤ upper bound

166 ≤ T ≤ 211

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7 The table below shows the least distances, in km, between six lighthouses, A to F, on the coast of Cornwall. An engineer must inspect every lighthouse and return to her start.

a. By deleting F, and using Prim's algorithm starting at A, find a lower bound for the length of her tour.

b. Show that the arcs you used in (a) form a tour, and state this tour.

c. Explain what you can deduce about this tour.

	1 A	5 B	2 C	3 D	4 E	F
A	-	43	16	33	52	60
B	43	-	41	27	13	58
C	16	41	-	31	48	61
D	33	27	31	-	23	67
E	52	13	48	23	-	65
F	60	58	61	67	65	-

Deleted vertex method

Deleted vertex	RMST weight	Two least arcs	Lower bound
F	83	FB 58, FA 60	201

(7 marks)

a)

Delete F and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 16 in column A, row C: add AC

Delete row C, number column C as 2

Smallest is 31 in column C, row D: add CD

Delete row D, number column D as 3

Smallest is 23 in column D, row E: add DE

Delete row E, number column E as 4

Smallest is 13 in column E, row B: add EB

Delete row B, number column B as 5

Weight = 16 + 31 + 23 + 13 = 83

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Arcs from F: FA 60, FB 58, FC 61, FD 67, FE 65

Two least: FB 58 and FA 60

Lower bound = 83 + 58 + 60 = 201

201 km

b)

The RMST is the path A - C - D - E - B (each vertex degree ≤ 2)

The two arcs from F go to B and A, the two ends of the path

Tour: F - A - C - D - E - B - F

Length = 60 + 16 + 31 + 23 + 13 + 58 = 201

F - A - C - D - E - B - F

c)

Any tour gives an upper bound: optimal ≤ 201

Lower bound: optimal ≥ 201

So optimal = 201

The tour is optimal, length 201 km

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8 The table shows the least distances, in miles, between six garden centres, A to F, in Hampshire. The least distance between A and C is x miles, where $52 < x < 65$

	A	B	C	D	E	F
A	–	54	x	14	47	55
B	54	–	72	53	18	12
C	x	72	–	65	74	84
D	14	53	65	–	35	41
E	47	18	74	35	–	10
F	55	12	84	41	10	–

a. By deleting C, find a lower bound, in terms of x, for the length of a tour of the garden centres.

b. By deleting B, find a second lower bound in terms of x

c. State which of the two lower bounds is better, giving a reason.

d. The tour A - C - B - F - E - D - A has length $(143 + x)$ miles. Write down an interval, in terms of x, that must contain the length, L, of the optimal tour.

Deleted vertex method

Deleted vertex	RMST weight	Two least arcs	Lower bound
C	71	CA x, CD 65	136 + x
B	59 + x	BF 12, BE 18	89 + x

(7 marks)

a)

Delete C and all of its arcs

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

Smallest is 14 in column A, row D: add AD

Delete row D, number column D as 2

Smallest is 35 in column D, row E: add DE

Delete row E, number column E as 3

Smallest is 10 in column E, row F: add EF

Delete row F, number column F as 4

Smallest is 12 in column F, row B: add FB

Delete row B, number column B as 5

Weight = 14 + 35 + 10 + 12 = 71

Arcs from C: CA x, CB 72, CD 65, CE 74, CF 84

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$x < 65$, so the two least are CA x and CO 65

Lower bound $= 71 + x + 65 = 136 + x$

$\boxed{136 + x}$

b)

Delete B and all of its arcs; Prim from A

Smallest is 14 in column A, row O: add AO

Columns A, O: C x or 65, E 35, F 41: add OE (35)

Columns A, O, E: C x or 65, F 10: add EF (10)

Last: C, smallest of x, 65, 74, 84 is x ($x < 65$): add AC

RMST weight $= 14 + 35 + 10 + x = 59 + x$

Arcs from B: BA 54, BC 72, BO 53, BE 18, BF 12

Two least: BF 12 and BE 18

Lower bound $= 59 + x + 12 + 18 = 89 + x$

$\boxed{89 + x}$

c)

$136 + x > 89 + x$ for every x

The best lower bound is the larger

$\boxed{136 + x}$

d)

Lower bound $\leqslant L \leqslant$ upper bound

$\boxed{136 + x \leqslant L \leqslant 143 + x}$

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9 The table shows the least distances, in km, between seven campsites, A to G, in the Lake District. A warden visits every campsite, starting and finishing at A.

	A	B	C	D	E	F	G
A	–	32	71	22	77	61	48
B	32	–	39	30	45	34	20
C	71	39	–	63	8	12	23
D	22	30	63	–	62	51	40
E	77	45	8	62	–	18	31
F	61	34	12	51	18	–	16
G	48	20	23	40	31	16	–

a. Use the nearest neighbour algorithm, starting at A, to find an upper bound for the length of the route.

b. By deleting A, find a lower bound for the length of the route.

c. By deleting B, find a second lower bound.

d. Using your answers, write down the smallest interval that must contain the length, L km, of the optimal route.

Upper bound

Start	Tour	Length
A	A - D - B - G - F - C - E - A	185

Lower bounds

Deleted vertex	RMST weight	Two least arcs	Lower bound
A	86	AD 22, AB 32	140
B	98	BG 20, BD 30	148

(9 marks)

a)

Nearest neighbour from A: each time go to the nearest vertex not yet visited

From A: D 22, B 32, G 48, F 61, C 71, E 77, nearest is D

From D: B 30, G 40, F 51, E 62, C 63, nearest is B

From B: G 20, F 34, C 39, E 45, nearest is G

From G: F 16, C 23, E 31, nearest is F

From F: C 12, E 18, nearest is C

From C: only E is left (8)

All vertices visited: return E to A (77)

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Tour: A - O - B - G - F - C - E - A

Length = 22 + 30 + 20 + 16 + 12 + 8 + 77 = 185

A - O - B - G - F - C - E - A, 185 km

b)

Delete A and all of its arcs

Find a minimum spanning tree of the remaining 6 vertices (Prim)

Arcs: BG 20, GF 16, FC 12, CE 8, BO 30

RMST weight = 20 + 16 + 12 + 8 + 30 = 86

Arcs from A: AB 32, AC 71, AO 22, AE 77, AF 61, AG 48

Two least: AO 22 and AB 32

Lower bound = 86 + 22 + 32 = 140

140 km

c)

Delete B and all of its arcs

Find a minimum spanning tree of the remaining 6 vertices (Prim)

Arcs: AO 22, OG 40, GF 16, FC 12, CE 8

RMST weight = 22 + 40 + 16 + 12 + 8 = 98

Arcs from B: BA 32, BC 39, BO 30, BE 45, BF 34, BG 20

Two least: BG 20 and BO 30

Lower bound = 98 + 20 + 30 = 148

148 km

d)

Best lower bound = larger = 148

Upper bound = 185

$148 \leq L \leq 185$

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10 The network below shows the cycle paths between six stations, A to F, in Cambridge. The numbers on the arcs are lengths in hundreds of metres.

a. Complete the table of least distances below.

b. By deleting A, find a lower bound for the length of a tour of the six stations.

c. By deleting C, find a second lower bound.

d. State the better lower bound, giving a reason.

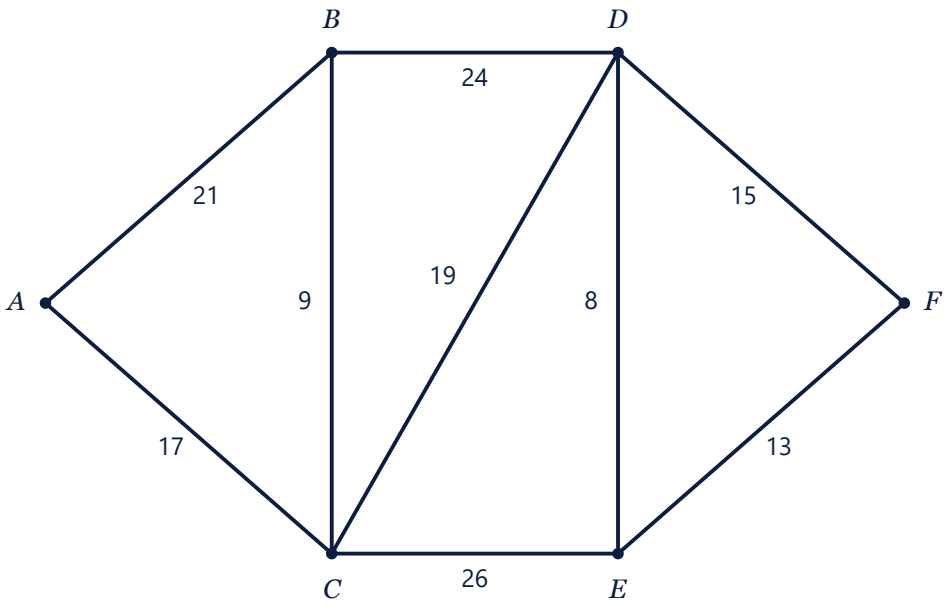


Table of least distances

	A	B	C	D	E	F
A	-	21	17	36	43	51
B	21	-	9	24	32	39
C	17	9	-	19	26	34
D	36	24	19	-	8	15
E	43	32	26	8	-	13
F	51	39	34	15	13	-

Lower bounds

Deleted vertex	RMST weight	Two least arcs	Lower bound
A	49	AC 17, AB 21	87
C	66	CB 9, CA 17	92

Diagram NOT accurately drawn

(8 marks)

a)

Shortest path for each blank by inspection

AD: A - C - D = 17 + 19 = 36

(A - B - D = 21 + 24 = 45 is longer)

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AF: A - C - O - F = 17 + 19 + 15 = 51

BF: B - O - F = 24 + 15 = 39

CF: C - O - F = 19 + 15 = 34

AO 36, AF 51, BF 39, CF 34

b)

Delete A and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: BC 9, CO 19, OE 8, EF 13

RMST weight = 9 + 19 + 8 + 13 = 49

Arcs from A: AB 21, AC 17, AO 36, AE 43, AF 51

Two least: AC 17 and AB 21

Lower bound = 49 + 17 + 21 = 87

87

c)

Delete C and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: AB 21, BO 24, OE 8, EF 13

RMST weight = 21 + 24 + 8 + 13 = 66

Arcs from C: CA 17, CB 9, CO 19, CE 26, CF 34

Two least: CB 9 and CA 17

Lower bound = 66 + 9 + 17 = 92

92

d)

The optimal tour $\geqslant$ 87 and $\geqslant$ 92

92 (9200 m) is the better lower bound

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11 The table shows the least distances, in km, between six farms, A to F, in Devon. The least distance between B and D is x km, where $36 < x < 49$

	A	B	C	D	E	F
A	–	8	21	44	17	19
B	8	–	13	x	14	11
C	21	13	–	49	27	16
D	44	x	49	–	35	47
E	17	14	27	35	–	25
F	19	11	16	47	25	–

a. By deleting A, find a lower bound for the length of a tour of the farms.

b. By deleting D, find a lower bound in terms of x when $x < 44$, and the lower bound when $x \geq 44$

c. Use the nearest neighbour algorithm, starting at A, to find an upper bound.

The better of the lower bounds in (a) and (b) is 120 km.

d. Find the value of x and write down an interval that must contain the length, L km, of the optimal tour.

Lower bounds

Deleted vertex	RMST weight	Two least arcs	Lower bound
A	73	AB 8, AE 17	98
D (x < 44)	46	DE 35, DB x	81 + x
D (x ≥ 44)	46	DE 35, DA 44	125

Upper bound

Start	Tour	Length
A	A - B - F - C - E - D - A	141

(10 marks)

a)

Delete A and all of its arcs

Find a minimum spanning tree of the remaining 5 vertices (Prim)

Arcs: BF 11, BC 13, BE 14, ED 35

RMST weight = 11 + 13 + 14 + 35 = 73

Arcs from A: AB 8, AC 21, AD 44, AE 17, AF 19

Two least: AB 8 and AE 17

Lower bound = 73 + 8 + 17 = 98

98 km

b)

Delete O and all of its arcs (x is not used in the RMST)

Smallest is 8 in column A, row B: add AB

Columns A, B: C 13, E 14, F 11: add BF (11)

Columns A, B, F: C 13, E 14: add BC (13)

Last: E 14: add BE

RMST weight = 8 + 11 + 13 + 14 = 46

Arcs from O: OA 44, OB x, OC 49, OE 35, OF 47

Case $x < 44$

Two least: OE 35 and OB x

Lower bound = 46 + 35 + x = 81 + x

Case $x \geqslant 44$

Two least: OE 35 and OA 44

Lower bound = 46 + 35 + 44 = 125

$81 + x$ ($x < 44$); 125 ($x \geqslant 44$)

c)

Nearest neighbour from A: each time go to the nearest vertex not yet visited

From A: B 8, E 17, F 19, C 21, O 44, nearest is B

From B: F 11, C 13, E 14, O x ($x > 36$), nearest is F

From F: C 16, E 25, O 47, nearest is C

From C: E 27, O 49, nearest is E

From E: only O is left (35)

All vertices visited: return O to A (44)

Tour: A - B - F - C - E - O - A

Length = 8 + 11 + 16 + 27 + 35 + 44 = 141

A - B - F - C - E - O - A, 141 km

d)

$120 \neq 98$ and $120 \neq 125$, so $120 = 81 + x$

$x = 120 - 81 = 39$

Check: $39 < 44$ and $81 + 39 = 120 > 98$

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$x = 39;\ 120 \leq L \leq 141$

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12 A heritage inspector must visit eight castles in Wales, A to H, starting and finishing at A. The table below shows the least driving distances between them, in miles.

a. By deleting B, and using Prim's algorithm starting at A, find a lower bound for the length of the inspector's route. State the order in which you select the arcs.

Deleting A gives a lower bound of 177 miles and deleting C gives a lower bound of 188 miles.

b. State the best lower bound, giving a reason.

c. Use the nearest neighbour algorithm, starting at A, to find an upper bound.

d. Write down the smallest interval that must contain the length, L miles, of the optimal route.

e. A colleague claims to have found a route of 198 miles. Explain why the colleague must be wrong.

	1		4	2	3	7	5	6
	A	**B**	**C**	**D**	**E**	**F**	**G**	**H**
A	-	43	34	13	14	50	33	45
B	43	-	64	56	51	11	48	68
C	34	64	-	21	39	70	67	76
D	13	56	21	-	18	59	46	55
E	14	51	39	18	-	57	32	37
F	50	11	70	59	57	-	58	66
G	33	48	67	46	32	58	-	20
H	45	68	76	55	37	66	20	-

Lower bounds

Deleted vertex	RMST weight	Two least arcs	Lower bound
B	150	BF 11, BA 43	204
A			177
C			188

Upper bound

Start	Tour	Length
A	A - D - E - G - H - F - B - C - A	258

(11 marks)

a)

Delete B and all of its arcs

Find a minimum spanning tree of the remaining 7 vertices (Prim)

Start at A: delete row A, number column A as 1

Each time find the smallest undeleted entry in the numbered columns

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Smallest is 13 in column A, row O: add AO

Delete row O, number column O as 2

Smallest is 14 in column A, row E: add AE

Delete row E, number column E as 3

Smallest is 21 in column O, row C: add OC

Delete row C, number column C as 4

Smallest is 32 in column E, row G: add EG

Delete row G, number column G as 5

Smallest is 20 in column G, row H: add GH

Delete row H, number column H as 6

Smallest is 50 in column A, row F: add AF

Delete row F, number column F as 7

Weight = 13 + 14 + 21 + 32 + 20 + 50 = 150

Arcs from B: BA 43, BC 64, BO 56, BE 51, BF 11, BG 48, BH 68

Two least: BF 11 and BA 43

Lower bound = 150 + 11 + 43 = 204

AO, AE, OC, EG, GH, AF; lower bound 204 miles

b)

Lower bounds: 204, 177, 188

The optimal route is at least every lower bound

204 miles (the largest)

c)

Nearest neighbour from A: each time go to the nearest vertex not yet visited

From A: O 13, E 14, G 33, C 34, B 43, H 45, F 50, nearest is O

From O: E 18, C 21, G 46, H 55, B 56, F 59, nearest is E

From E: G 32, H 37, C 39, B 51, F 57, nearest is G

From G: H 20, B 48, F 58, C 67, nearest is H

From H: F 66, B 68, C 76, nearest is F

From F: B 11, C 70, nearest is B

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From B: only C is left (64)

All vertices visited: return C to A (34)

Tour: A - D - E - G - H - F - B - C - A

Length = 13 + 18 + 32 + 20 + 66 + 11 + 64 + 34 = 258

A - D - E - G - H - F - B - C - A, 258 miles

d)

$204 \leqslant L \leqslant 258$

e)

Every route has length $\geqslant 204$

$198 < 204$

198 is below the lower bound, so it is impossible

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