Two-stage simplex and big-M

Worked solutions

Further Maths topic practice

This topic online

1 A linear programming problem is given as

Maximise $P = 3x + 4y$

subject to

$x + 2y \leq 20$

$x + y \geq 6$

$x, y \geq 0$

a. Explain why the Simplex algorithm cannot be used directly to solve this problem.

b. Write each constraint as an equation, using exactly one slack variable, one surplus variable and one artificial variable. State which of your variables is which.

c. Explain why the artificial variable is needed.

(4 marks)

a)

At the origin $x + y = 0$, which is not ≥ 6

The origin is not feasible, so there is no starting vertex

b)

$\leq$ constraint: add a slack variable

$x + 2y + s1 = 20$

$\geq$ constraint: subtract a surplus, add an artificial

$x + y - s2 + a1 = 6$

s1 slack, s2 surplus, a1 artificial (all ≥ 0)

c)

Start with $x = y = 0$

$-s2 = 6$ without a1, so $s2 = -6$

s2 must be ≥ 0, so this is not allowed

a1 = 6 gives a basic feasible starting solution

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2 A linear programming problem is given as

Maximise $P = 2x + 3y + z$

subject to

$x + y + z \leq 30$

$2x + y \geq 8$

$x, y, z \geq 0$

The problem is to be solved using the big-M method, with slack variable s_1, surplus variable s_2 and artificial variable a_1.

a. Define, in this context, what M represents.

b. Show that the objective row of the initial tableau represents the equation

$P - (2 + 2M)x - (3 + M)y - z + Ms_2 = -8M$

(4 marks)

a)

M is an arbitrarily large positive real number

b)

Constraint equations

$x + y + z + s1 = 30$

$2x + y - s2 + a1 = 8$

Penalise the artificial variable in the objective

$P = 2x + 3y + z - Ma1$

Make a1 the subject

$a1 = 8 - 2x - y + s2$

Substitute

$P = 2x + 3y + z - M(8 - 2x - y + s2)$

$P = 2x + 3y + z - 8M + 2Mx + My - Ms2$

$P = (2 + 2M)x + (3 + M)y + z - Ms2 - 8M$

Every term in x, y, z, s2 to the left

$P - (2 + 2M)x - (3 + M)y - z + Ms2 = -8M$

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3 A linear programming problem is given as

Maximise $P = 5x + 3y + 2z$

subject to

$3x + y + 2z \leq 40$

$x + y + z \geq 10$

$y - z \geq 2$

$x, y, z \geq 0$

The problem is to be solved using the big-M method.

a. Using slack variable s_1, surplus variables s_2 and s_3, and artificial variables a_1 and a_2, set up the initial tableau. You must show how you obtain the objective row. Use the template below.

b. Identify the pivot for the first iteration of the big-M method, giving reasons for your choice.

Initial tableau

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
s_1	3	1	2	1	0	0	0	0	40
a_1	1	1	1	0	−1	0	1	0	10
a_2	0	1	−1	0	0	−1	0	1	2
P	−5 − M	−3 − 2M	−2	0	M	M	0	0	−12M

(7 marks)

a)

Constraint equations

$$3x + y + 2z + s1 = 40$$

$$x + y + z - s2 + a1 = 10$$

$$y - z - s3 + a2 = 2$$

$$P = 5x + 3y + 2z - Ma1 - Ma2$$

$$a1 + a2 = 12 - x - 2y + s2 + s3$$

$$P = 5x + 3y + 2z - M(12 - x - 2y + s2 + s3)$$

$$P - (5 + M)x - (3 + 2M)y - 2z + Ms2 + Ms3 = -12M$$

Initial tableau (columns: x, y, z, s1, s2, s3, a1, a2 | Value)

s1: 3, 1, 2, 1, 0, 0, 0, 0 | 40

a1: 1, 1, 1, 0, −1, 0, 1, 0 | 10

a2: 0, 1, −1, 0, 0, −1, 0, 1 | 2

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P: $-5 - M, -3 - 2M, -2, 0, M, M, 0, 0 \mid -12M$

Initial tableau as shown

b)

Compare the M terms: $-2M$ is the most negative coefficient of M

Most negative entry in the P row is $-3 - 2M$ (the y column, as M is very large): y enters

θ for R1: $40 \div 1 = 40$

θ for R2: $10 \div 1 = 10$

θ for R3: $2 \div 1 = 2$

Smallest θ is 2 in R3: pivot is 1, y replaces a2

Pivot is the 1 in the a2 row, y column

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4 A linear programming problem is given as

Maximise $P = 4x + y + 3z$

subject to

$x + 2y + z \leq 30$

$2x + y \geq 10$

$x + y + z = 15$

$x, y, z \geq 0$

The problem is to be solved using the two-stage Simplex method.

a. Write the three constraints as equations, using slack variable s_1, surplus variable s_2 and artificial variables a_1 and a_2.

b. Set up the initial tableau for the first stage, including the objective row for P and the row for the new objective A. You should make your method for the A row clear. Use the template below.

Initial tableau

b.v.	x	y	z	s_1	s_2	a_1	a_2	Value
s_1	1	2	1	1	0	0	0	30
a_1	2	1	0	0	−1	1	0	10
a_2	1	1	1	0	0	0	1	15
P	−4	−1	−3	0	0	0	0	0
A	−3	−2	−1	0	1	0	0	−25

(7 marks)

a)

$x + 2y + z + s1 = 30$

$2x + y - s2 + a1 = 10$

$x + y + z + a2 = 15$

An equation needs no slack or surplus, only an artificial variable

$x + 2y + z + s1 = 30,\ 2x + y - s2 + a1 = 10,\ x + y + z + a2 = 15$

b)

Stage 1: minimise A = a1 + a2

$A = (10 - 2x - y + s2) + (15 - x - y - z)$

$A = 25 - 3x - 2y - z + s2$

Write the A row with every variable on the left (maximising -A)

$-A - 3x - 2y - z + s2 = -25$

P row from $P - 4x - y - 3z = 0$

Initial tableau (columns: x, y, z, s1, s2, a1, a2 | Value)

s1: 1, 2, 1, 1, 0, 0, 0 | 30

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a1: 2, 1, 0, 0, -1, 1, 0 | 10

a2: 1, 1, 1, 0, 0, 0, 1 | 15

P: -4, -1, -3, 0, 0, 0, 0 | 0

A: -3, -2, -1, 0, 1, 0, 0 | -25

Initial tableau as shown, with both a P row and an A row

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5 A linear programming problem in x, y and z is to be solved using the big-M method. The initial tableau is shown below.

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
s_1	1	2	1	1	0	0	0	0	24
a_1	1	−1	2	0	−1	0	1	0	6
a_2	3	1	0	0	0	−1	0	1	9
P	−2 − 4M	1	−3 − 2M	0	M	M	0	0	−15M

a. Using the information in the tableau, formulate the linear programming problem. You should list each constraint as an inequality and state two possible ways of writing the objective.

b. Obtain the pivot for the first iteration of the big-M method. You must give reasons for your answer.

(7 marks)

a)

Row s1: slack +1, so ≤

$x + 2y + z + s1 = 24$

$x + 2y + z \leq 24$

Rows a1 and a2: surplus −1 and an artificial, so ≥

$x - y + 2z - s2 + a1 = 6$

$x - y + 2z \geq 6$

$3x + y - s3 + a2 = 9$

$3x + y \geq 9$

P row = (P row without M) − M × (a1 row + a2 row)

a1 row + a2 row: $4x + 0y + 2z - s2 - s3 = 15$

P row: $-2 - 4M, 1, -3 - 2M, 0, M, M, 0, 0 \mid -15M$

Remove $-M \times (4, 0, 2, 0, -1, -1 \mid 15)$

$P - 2x + y - 3z = 0$

Maximise $P = 2x - y + 3z$, or minimise $C = -2x + y - 3z$, subject to $x + 2y + z \leq 24$, $x - y + 2z \geq 6$, $3x + y \geq 9$, $x, y, z \geq 0$

b)

M terms: x has −4M, z has −2M; −4M is the most negative

Most negative entry in the P row is −2 − 4M (the x column, as M is very large): x enters

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θ for R1: $24 \div 1 = 24$

θ for R2: $6 \div 1 = 6$

θ for R3: $9 \div 3 = 3$

Smallest θ is 3 in R3: pivot is 3, x replaces a2

The 3 in the a2 row (x column)

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6 A maximisation linear programming problem in x, y and z is to be solved using the two-stage Simplex method. The partially completed initial tableau is shown in the template below; the objective for the first stage is to minimise $A = a_1 + a_2$.

a. Complete the bottom row of the initial tableau. You should make your method clear.

b. Carry out the first stage of the two-stage Simplex method, taking the most negative number in the A row to indicate the pivot column each time. State the row operations you use.

c. State the basic feasible solution found at the end of the first stage.

Initial tableau

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row ops
s_1	2	1	2	1	0	0	0	0	23	
a_1	2	0	1	0	−1	0	1	0	12	
a_2	3	2	1	0	0	−1	0	1	16	
P	−1	−4	−1	0	0	0	0	0	0	
A	−5	−2	−2	0	1	1	0	0	−28	

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row ops
s_1	0	$-\frac{1}{3}$	$\frac{4}{3}$	1	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{37}{3}$	R1 − 2R3
a_1	0	$-\frac{4}{3}$	$\frac{1}{3}$	0	−1	$\frac{2}{3}$	1	$-\frac{2}{3}$	$\frac{4}{3}$	R2 − 2R3
x	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{16}{3}$	R3 ÷ 3
P	0	$-\frac{10}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{16}{3}$	R4 + R3
A	0	$\frac{4}{3}$	$-\frac{1}{3}$	0	1	$-\frac{2}{3}$	0	$\frac{5}{3}$	$-\frac{4}{3}$	R5 + 5R3

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value	Row ops
s_1	0	1	1	1	1	0	−1	0	11	R1 − 2/3 R2
s_3	0	−2	$\frac{1}{2}$	0	$-\frac{3}{2}$	1	$\frac{3}{2}$	−1	2	R2 × 3/2
x	1	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	6	R3 + 1/3 R2
P	0	−4	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	6	R4 + 1/3 R2
A	0	0	0	0	0	0	1	1	0	R5 + 2/3 R2

(9 marks)

a)

$A = a1 + a2$

$a1 = 12 - 2x - z + s2$

$a2 = 16 - 3x - 2y - z + s3$

$A = 28 - 5x - 2y - 2z + s2 + s3$

$-A - 5x - 2y - 2z + s2 + s3 = -28$

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A row: $-5, -2, -2, 0, 1, 1, 0, 0 \mid -28$

Initial tableau (columns: $x, y, z, s1, s2, s3, a1, a2 \mid$ Value)

$s1: 2, 1, 2, 1, 0, 0, 0, 0 \mid 23$

$a1: 2, 0, 1, 0, -1, 0, 1, 0 \mid 12$

$a2: 3, 2, 1, 0, 0, -1, 0, 1 \mid 16$

$P: -1, -4, -1, 0, 0, 0, 0, 0 \mid 0$

$A: -5, -2, -2, 0, 1, 1, 0, 0 \mid -28$

A row: $-5, -2, -2, 0, 1, 1, 0, 0$	-28

b)

Stage 1: choose the pivot column from the A row

Most negative entry in the A row is -5, in the x column: x enters

θ for R1: $23 \div 2 = \frac{23}{2}$

θ for R2: $12 \div 2 = 6$

θ for R3: $16 \div 3 = \frac{16}{3}$

Smallest θ is $\frac{16}{3}$ in R3: pivot is 3, x replaces a2

Divide the pivot row by the pivot, then clear the rest of the x column using the new R3

Tableau after 1st iteration (columns: $x, y, z, s1, s2, s3, a1, a2 \mid$ Value $\mid$ Row ops)

$s1: 0, -\frac{1}{3}, \frac{4}{3}, 1, 0, \frac{2}{3}, 0, -\frac{2}{3} \mid \frac{37}{3} \mid R1 - 2R3$

$a1: 0, -\frac{4}{3}, \frac{1}{3}, 0, -1, \frac{2}{3}, 1, -\frac{2}{3} \mid \frac{4}{3} \mid R2 - 2R3$

$x: 1, \frac{2}{3}, \frac{1}{3}, 0, 0, -\frac{1}{3}, 0, \frac{1}{3} \mid \frac{16}{3} \mid R3 \div 3$

$P: 0, -\frac{10}{3}, -\frac{2}{3}, 0, 0, -\frac{1}{3}, 0, \frac{1}{3} \mid \frac{16}{3} \mid R4 + R3$

$A: 0, \frac{4}{3}, -\frac{1}{3}, 0, 1, -\frac{2}{3}, 0, \frac{5}{3} \mid -\frac{4}{3} \mid R5 + 5R3$

Most negative entry in the A row is $-\frac{2}{3}$, in the s3 column: s3 enters

θ for R1: $\frac{37}{3} \div \frac{2}{3} = \frac{37}{2}$

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θ for R2: $\frac{4}{3} \div \frac{2}{3} = 2$

R3: entry not positive, no θ

Smallest θ is 2 in R2: pivot is $\frac{2}{3}$, s3 replaces a1

Tableau after 2nd iteration (columns: x, y, z, s1, s2, s3, a1, a2 | Value | Row ops)

s1: $0, 1, 1, 1, 1, 0, -1, 0 \mid 11 \mid R1 - \frac{2}{3}R2$

s3: $0, -2, \frac{1}{2}, 0, -\frac{3}{2}, 1, \frac{3}{2}, -1 \mid 2 \mid R2 \times \frac{3}{2}$

x: $1, 0, \frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0 \mid 6 \mid R3 + \frac{1}{3}R2$

P: $0, -4, -\frac{1}{2}, 0, -\frac{1}{2}, 0, \frac{1}{2}, 0 \mid 6 \mid R4 + \frac{1}{3}R2$

A: $0, 0, 0, 0, 0, 0, 1, 1 \mid 0 \mid R5 + \frac{2}{3}R2$

A row value is 0, so A = 0: the first stage is complete

c)

The artificial variables are non-basic, so zero

Basic: s1 = 11, s3 = 2, x = 6

Non-basic: y = 0, z = 0, s2 = 0, a1 = 0, a2 = 0

x = 6, y = 0, z = 0, s1 = 11, s2 = 0, s3 = 2 (P = 6)

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7 A linear programming problem is given as

Maximise $P = 6x + 4y$

subject to

$x + y \leq 12$

$x + 2y \leq 14$

$y \geq 4$

$x, y \geq 0$

a. Using slack variables s_1 and s_2, surplus variable s_3 and artificial variable a_1, set up the initial tableau for solving this problem using the big-M method.

b. Solve the problem using the big-M method, taking the most negative number in the objective row to indicate the pivot column each time. State the row operations you use and the optimal values of x, y and P.

Use the templates below.

Initial tableau

b.v.	x	y	s_1	s_2	s_3	a_1	Value	Row ops
s_1	1	1	1	0	0	0	12	
s_2	1	2	0	1	0	0	14	
a_1	0	1	0	0	−1	1	4	
P	−6	−4 − M	0	0	M	0	−4M	

Tableau after 1st iteration

b.v.	x	y	s_1	s_2	s_3	a_1	Value	Row ops
s_1	1	0	1	0	1	−1	8	R1 − R3
s_2	1	0	0	1	2	−2	6	R2 − 2R3
y	0	1	0	0	−1	1	4	R3 ÷ 1
P	−6	0	0	0	−4	4 + M	16	R4 + (4 + M)R3

Tableau after 2nd iteration

b.v.	x	y	s_1	s_2	s_3	a_1	Value	Row ops
s_1	0	0	1	−1	−1	1	2	R1 − R2
x	1	0	0	1	2	−2	6	R2 ÷ 1
y	0	1	0	0	−1	1	4	R3
P	0	0	0	6	8	−8 + M	52	R4 + 6R2

(10 marks)

a)

$$x + y + s1 = 12$$

$$x + 2y + s2 = 14$$

$$y - s3 + a1 = 4$$

$$P = 6x + 4y - Ma1$$

$$P = 6x + 4y - M(4 - y + s3)$$

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$P - 6x - (4 + M)y + Ms3 = -4M$

Initial tableau (columns: x, y, s1, s2, s3, a1 | Value)

s1: 1, 1, 1, 0, 0, 0 | 12

s2: 1, 2, 0, 1, 0, 0 | 14

a1: 0, 1, 0, 0, -1, 1 | 4

P: -6, -4 - M, 0, 0, M, 0 | -4M

b)

Most negative entry in the P row is -4 - M (the y column, as M is very large): y enters

θ for R1: $12 \div 1 = 12$

θ for R2: $14 \div 2 = 7$

θ for R3: $4 \div 1 = 4$

Smallest θ is 4 in R3: pivot is 1, y replaces a1

Divide the pivot row by the pivot, then clear the rest of the y column using the new R3

Tableau after 1st iteration (columns: x, y, s1, s2, s3, a1 | Value | Row ops)

s1: 1, 0, 1, 0, 1, -1 | 8 | R1 - R3

s2: 1, 0, 0, 1, 2, -2 | 6 | R2 - 2R3

y: 0, 1, 0, 0, -1, 1 | 4 | R3 ÷ 1

P: -6, 0, 0, 0, -4, 4 + M | 16 | R4 + (4 + M)R3

Most negative entry in the P row is -6, in the x column: x enters

θ for R1: $8 \div 1 = 8$

θ for R2: $6 \div 1 = 6$

R3: entry not positive, no θ

Smallest θ is 6 in R2: pivot is 1, x replaces s2

Tableau after 2nd iteration (columns: x, y, s1, s2, s3, a1 | Value | Row ops)

s1: 0, 0, 1, -1, -1, 1 | 2 | R1 - R2

x: 1, 0, 0, 1, 2, -2 | 6 | R2 ÷ 1

y: 0, 1, 0, 0, -1, 1 | 4 | R3

P: 0, 0, 0, 6, 8, -8 + M | 52 | R4 + 6R2

No negative values in the P row, so optimal

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Basic: $s1 = 2, x = 6, y = 4$

$a1$ is non-basic, so $a1 = 0$ and the solution is feasible

$$x = 6, y = 4, P = 52$$

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8 A linear programming problem in x and y is being solved using the big-M method. The constraints are $3x + y \leq 35$, $x + y \geq 7$ and $x \geq 4$, and the objective is to maximise $P = 3x + 4y$. After the first iteration the tableau is

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value
s_1	0	1	1	0	3	0	−3	23
a_1	0	1	0	−1	1	1	−1	3
x	1	0	0	0	−1	0	1	4
P	0	$-4 - M$	0	M	$-3 - M$	0	3 + 2M	12 − 3M

a. Explain why the solution given by this tableau is not feasible.

b. State the values of x and y given by this tableau.

c. Identify the pivot for the second iteration. You must give reasons for your answer.

d. Perform the second iteration, stating the row operations you use. Use the template below.

e. Explain whether the tableau you obtained in part (d) gives a feasible solution.

Tableau after 2nd iteration

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row ops
s_1	0	0	1	1	2	−1	−2	20	R1 − R2
y	0	1	0	−1	1	1	−1	3	R2 ÷ 1
x	1	0	0	0	−1	0	1	4	R3
P	0	0	0	−4	1	4 + M	−1 + M	24	R4 + (4 + M)R2

(9 marks)

a)

a1 is basic with value 3

Not feasible: an artificial variable is non-zero

b)

Basic: s1 = 23, a1 = 3, x = 4

Non-basic: y = 0, s2 = 0, s3 = 0, a2 = 0

x = 4, y = 0

c)

Compare M terms in the P row: -M appears in the y and s3 columns

-4 - M is more negative than -3 - M

Most negative entry in the P row is -4 - M (the y column, as M is very large): y enters

θ for R1: 23 ÷ 1 = 23

θ for R2: 3 ÷ 1 = 3

R3: entry not positive, no θ

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Smallest θ is 3 in R2: pivot is 1, y replaces a1

Pivot is the 1 in the a1 row (y column)

d)

Tableau after 2nd iteration (columns: x, y, s1, s2, s3, a1, a2 | Value | Row ops)

s1: 0, 0, 1, 1, 2, -1, -2 | 20 | R1 − R2

y: 0, 1, 0, −1, 1, 1, −1 | 3 | R2 ÷ 1

x: 1, 0, 0, 0, −1, 0, 1 | 4 | R3

P: 0, 0, 0, −4, 1, 4 + M, −1 + M | 24 | R4 + (4 + M)R2

e)

Both artificial variables are now non-basic, so zero

Feasible: $x = 4$, $y = 3$, $P = 24$

(Not yet optimal: there is still −4 in the s2 column)

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9 A linear programming problem is given as

Maximise $P = 6x + y + 5z$

subject to

$2x + 4y \leq 38$

$x + 4y + z \leq 38$

$y \geq 5$

$x, y, z \geq 0$

It is being solved using the two-stage Simplex method, with slack variables s_1 and s_2, surplus variable s_3 and artificial variable a_1. The tableau at the end of the first stage is

b.v.	x	y	z	s_1	s_2	s_3	a_1	**Value**
s_1	2	0	0	1	0	4	−4	18
s_2	1	0	1	0	1	4	−4	18
y	0	1	0	0	0	−1	1	5
P	−6	0	−5	0	0	−1	1	5
A	0	0	0	0	0	0	1	0

a. Explain how the tableau shows that a basic feasible solution has been found.

b. Write down the initial tableau for the second stage and complete the second stage, taking the most negative number in the profit row to indicate the pivot column each time. State the row operations you use. Use the templates below.

c. State the optimal values of x, y, z and P.

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(9 marks)

Second stage initial tableau

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
s_1	2	0	0	1	0	4	18	
s_2	1	0	1	0	1	4	18	
y	0	1	0	0	0	−1	5	
P	−6	0	−5	0	0	−1	5	

Tableau after 1st iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
x	1	0	0	$\frac{1}{2}$	0	2	9	R1 ÷ 2
s_2	0	0	1	$-\frac{1}{2}$	1	2	9	R2 − R1
y	0	1	0	0	0	−1	5	R3
P	0	0	−5	3	0	11	59	R4 + 6R1

Tableau after 2nd iteration

b.v.	x	y	z	s_1	s_2	s_3	Value	Row ops
x	1	0	0	$\frac{1}{2}$	0	2	9	R1
z	0	0	1	$-\frac{1}{2}$	1	2	9	R2 ÷ 1
y	0	1	0	0	0	−1	5	R3
P	0	0	0	$\frac{1}{2}$	5	21	104	R4 + 5R2

a)

The value in the A row is 0, so A = 0

A = 0: every artificial variable is zero, so a basic feasible solution has been found

b)

Stage 2: delete the A row and the a1 column, then optimise P

Second stage initial tableau (columns: x, y, z, s1, s2, s3 | Value)

s1: 2, 0, 0, 1, 0, 4 | 18

s2: 1, 0, 1, 0, 1, 4 | 18

y: 0, 1, 0, 0, 0, −1 | 5

P: −6, 0, −5, 0, 0, −1 | 5

Most negative entry in the P row is −6, in the x column: x enters

θ for R1: 18 ÷ 2 = 9

θ for R2: 18 ÷ 1 = 18

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R3: entry not positive, no θ

Smallest θ is 9 in R1: pivot is 2, x replaces $s1$

Divide the pivot row by the pivot, then clear the rest of the x column using the new R1

Tableau after 1st iteration (columns: x, y, z, $s1$, $s2$, $s3$ | Value | Row ops)

x: $1, 0, 0, \frac{1}{2}, 0, 2 \mid 9 \mid R1 \div 2$

$s2$: $0, 0, 1, -\frac{1}{2}, 1, 2 \mid 9 \mid R2 - R1$

y: $0, 1, 0, 0, 0, -1 \mid 5 \mid R3$

P: $0, 0, -5, 3, 0, 11 \mid 59 \mid R4 + 6R1$

Most negative entry in the P row is -5, in the z column: z enters

θ for R2: $9 \div 1 = 9$

R1, R3: entry not positive, no θ

Smallest θ is 9 in R2: pivot is 1, z replaces $s2$

Tableau after 2nd iteration (columns: x, y, z, $s1$, $s2$, $s3$ | Value | Row ops)

x: $1, 0, 0, \frac{1}{2}, 0, 2 \mid 9 \mid R1$

z: $0, 0, 1, -\frac{1}{2}, 1, 2 \mid 9 \mid R2 \div 1$

y: $0, 1, 0, 0, 0, -1 \mid 5 \mid R3$

P: $0, 0, 0, \frac{1}{2}, 5, 21 \mid 104 \mid R4 + 5R2$

c)

No negative values in the P row, so optimal

Basic: $x = 9, z = 9, y = 5$

$$\boxed{x = 9, y = 5, z = 9, P = 104}$$

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10 A park manager mixes two fertilisers, Growfast and Bloomwell, for the flower beds. Each bag of Growfast contains 2 units of nitrogen and 1 unit of potash and costs £4. Each bag of Bloomwell contains 1 unit of nitrogen and 2 units of potash and costs £3. The mixture must contain at least 10 units of nitrogen and at least 11 units of potash, and there is room to store at most 12 bags.

The manager buys x bags of Growfast and y bags of Bloomwell and wants to minimise the cost, £C.

a. Formulate this as a linear programming problem in which the objective is to maximise P.

b. Set up the initial tableau for solving the problem using the big-M method, using surplus variables s_1 and s_2, slack variable s_3 and artificial variables a_1 and a_2. You must show how you obtain the objective row.

c. Solve the problem using the big-M method, stating the row operations you use. State how many bags of each fertiliser should be bought and the minimum cost.

Use the templates below.

Initial tableau

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row ops
a_1	2	1	−1	0	0	1	0	10	
a_2	1	2	0	−1	0	0	1	11	
s_3	1	1	0	0	1	0	0	12	
P	4 − 3M	3 − 3M	M	M	0	0	0	−21M	

Tableau after 1st iteration

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row ops
a_1	$\frac{3}{2}$	0	−1	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{9}{2}$	R1 − R2
y	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{11}{2}$	R2 ÷ 2
s_3	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{13}{2}$	R3 − R2
P	$\frac{5-3M}{2}$	0	M	$\frac{3-M}{2}$	0	0	$\frac{-3+3M}{2}$	$\frac{-33-9M}{2}$	R4 + (−3 + 3M)R2

Tableau after 2nd iteration

b.v.	x	y	s_1	s_2	s_3	a_1	a_2	Value	Row ops
x	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	3	R1 × 2/3
y	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	4	R2 − 1/2 R1
s_3	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$	$-\frac{1}{3}$	5	R3 − 1/2 R1
P	0	0	$\frac{5}{3}$	$\frac{2}{3}$	0	$\frac{-5+3M}{3}$	$\frac{-2+3M}{3}$	−24	R4 + (−5/2 + 3/2 M)R1

(12 marks)

a)

Nitrogen: $2x + y \geqslant 10$

Potash: $x + 2y \geqslant 11$

Storage: $x + y \leqslant 12$

$C = 4x + 3y$

Minimise C by maximising $P = -C$

$P = -4x - 3y$

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Maximise $P = -4x - 3y$ subject to $2x + y \geqslant 10$, $x + 2y \geqslant 11$, $x + y \leqslant 12$, $x, y \geqslant 0$

b)

$$2x + y - s1 + a1 = 10$$
$$x + 2y - s2 + a2 = 11$$
$$x + y + s3 = 12$$
$$P = -4x - 3y - Ma1 - Ma2$$
$$a1 + a2 = 21 - 3x - 3y + s1 + s2$$
$$P + (4 - 3M)x + (3 - 3M)y + Ms1 + Ms2 = -21M$$

Initial tableau (columns: x, y, s1, s2, s3, a1, a2 | Value)

a1: 2, 1, −1, 0, 0, 1, 0 | 10

a2: 1, 2, 0, −1, 0, 0, 1 | 11

s3: 1, 1, 0, 0, 1, 0, 0 | 12

P: 4 − 3M, 3 − 3M, M, M, 0, 0, 0 | −21M

c)

Most negative entry in the P row is 3 − 3M (the y column, as M is very large): y enters

θ for R1: $10 \div 1 = 10$

θ for R2: $11 \div 2 = \frac{11}{2}$

θ for R3: $12 \div 1 = 12$

Smallest θ is $\frac{11}{2}$ in R2: pivot is 2, y replaces a2

Divide the pivot row by the pivot, then clear the rest of the y column using the new R2

Tableau after 1st iteration (columns: x, y, s1, s2, s3, a1, a2 | Value | Row ops)

a1: $\frac{3}{2}$, 0, −1, $\frac{1}{2}$, 0, 1, $-\frac{1}{2}$ | $\frac{9}{2}$ | R1 − R2

y: $\frac{1}{2}$, 1, 0, $-\frac{1}{2}$, 0, 0, $\frac{1}{2}$ | $\frac{11}{2}$ | R2 ÷ 2

s3: $\frac{1}{2}$, 0, 0, $\frac{1}{2}$, 1, 0, $-\frac{1}{2}$ | $\frac{13}{2}$ | R3 − R2

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P: $\frac{5-3M}{2}$, 0, M, $\frac{3-M}{2}$, 0, 0, $\frac{-3+3M}{2}$ | $\frac{-33-9M}{2}$ | R4 + (-3 + 3M)R2

Most negative entry in the P row is $\frac{5-3M}{2}$ (the x column, as M is very large): x enters

θ for R1: $\frac{9}{2} \div \frac{3}{2} = 3$

θ for R2: $\frac{11}{2} \div \frac{1}{2} = 11$

θ for R3: $\frac{13}{2} \div \frac{1}{2} = 13$

Smallest θ is 3 in R1: pivot is $\frac{3}{2}$, x replaces a1

Tableau after 2nd iteration (columns: x, y, s1, s2, s3, a1, a2 | Value | Row ops)

x: 1, 0, $-\frac{2}{3}$, $\frac{1}{3}$, 0, $\frac{2}{3}$, $-\frac{1}{3}$ | 3 | R1 × $\frac{2}{3}$

y: 0, 1, $\frac{1}{3}$, $-\frac{2}{3}$, 0, $-\frac{1}{3}$, $\frac{2}{3}$ | 4 | R2 - $\frac{1}{2}$R1

s3: 0, 0, $\frac{1}{3}$, $\frac{1}{3}$, 1, $-\frac{1}{3}$, $-\frac{1}{3}$ | 5 | R3 - $\frac{1}{2}$R1

P: 0, 0, $\frac{5}{3}$, $\frac{2}{3}$, 0, $-\frac{5}{3}+M$, $-\frac{2}{3}+M$ | -24 | R4 + ($-\frac{5}{2}+\frac{3}{2}$M)R1

No negative values in the P row, so optimal

Basic: $x = 3$, $y = 4$, $s3 = 5$

$P = -24$

$C = -P = 24$

3 bags of Growfast and 4 bags of Bloomwell: minimum cost £24

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11 A linear programming problem is given as

Maximise $P = 2x + 3y$

subject to

$x + y \leq 6$

$x + 2y \geq 16$

$x, y \geq 0$

a. Using slack variable s_1, surplus variable s_2 and artificial variable a_1, set up the initial tableau for the first stage of the two-stage Simplex method, including the A row.

b. Carry out the first stage, stating the row operations you use, and explain what your final tableau shows about the problem.

c. Explain, by considering the constraints, why this result is correct.

Use the templates below.

Initial tableau

b.v.	x	y	s_1	s_2	a_1	Value	Row ops
s_1	1	1	1	0	0	6	
a_1	1	2	0	−1	1	16	
P	−2	−3	0	0	0	0	
A	−1	−2	0	1	0	−16	

Tableau after 1st iteration

b.v.	x	y	s_1	s_2	a_1	Value	Row ops
y	1	1	1	0	0	6	R1 ÷ 1
a_1	−1	0	−2	−1	1	4	R2 − 2R1
P	1	0	3	0	0	18	R3 + 3R1
A	1	0	2	1	0	−4	R4 + 2R1

(8 marks)

a)

$$x + y + s1 = 6$$

$$x + 2y - s2 + a1 = 16$$

$$A = a1 = 16 - x - 2y + s2$$

$$-A - x - 2y + s2 = -16$$

Initial tableau (columns: x, y, s1, s2, a1 | Value)

s1: 1, 1, 1, 0, 0 | 6

a1: 1, 2, 0, −1, 1 | 16

P: −2, −3, 0, 0, 0 | 0

A: −1, −2, 0, 1, 0 | −16

b)

Most negative entry in the A row is −2, in the y column: y enters

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θ for R1: $6 \div 1 = 6$

θ for R2: $16 \div 2 = 8$

Smallest θ is 6 in R1: pivot is 1, y replaces s1

Divide the pivot row by the pivot, then clear the rest of the y column using the new R1

Tableau after 1st iteration (columns: x, y, s1, s2, a1 | Value | Row ops)

y: 1, 1, 1, 0, 0 | 6 | R1 ÷ 1

a1: −1, 0, −2, −1, 1 | 4 | R2 − 2R1

P: 1, 0, 3, 0, 0 | 18 | R3 + 3R1

A: 1, 0, 2, 1, 0 | −4 | R4 + 2R1

No negative values remain in the A row, so the first stage has ended

$A = 4$

$A \neq 0$ ($a1 = 4$), so the problem has no feasible solution

c)

$x + y \leqslant 6$ with $x, y \geqslant 0$ means $x + 2y \leqslant 2(x + y) \leqslant 12$

So $x + 2y$ can be at most 12, never 16

The two constraints cannot both hold: the feasible region is empty

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12 An animal sanctuary mixes three dry feeds, Alfa, Bran and Corn. Each kilogram of Alfa contains 2 units of protein and costs £4, each kilogram of Bran contains 4 units of protein and costs £5, and each kilogram of Corn contains 3 units of protein and costs £4. Each day the sanctuary needs at least 40 kg of feed containing at least 130 units of protein in total. It can store at most 25 kg of Alfa.

The sanctuary uses x kg of Alfa, y kg of Bran and z kg of Corn each day, and wants to minimise the daily cost, £C.

a. Formulate this as a linear programming problem.

b. The problem is to be solved by maximising $P = -C$ using the big-M method, with surplus variables s_1 and s_2, slack variable s_3 and artificial variables a_1 and a_2. Set up the initial tableau, showing how you obtain the objective row. Use the template below.

After three iterations of the big-M method the tableau is

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
z	2	0	1	−4	1	0	4	−1	30
y	−1	1	0	3	−1	0	−3	1	10
s_3	1	0	0	0	0	1	0	0	25
P	1	0	0	1	1	0	$-1 + M$	$-1 + M$	−170

c. Explain how you know this tableau is optimal.

d. State how much of each feed the sanctuary should use each day and the minimum daily cost.

e. Interpret the values of the slack and surplus variables in context.

Initial tableau

b.v.	x	y	z	s_1	s_2	s_3	a_1	a_2	Value
a_1	1	1	1	−1	0	0	1	0	40
a_2	2	4	3	0	−1	0	0	1	130
s_3	1	0	0	0	0	1	0	0	25
P	4 − 3M	5 − 5M	4 − 4M	M	M	0	0	0	−170M

(12 marks)

a)

Total mass: $x + y + z \geq 40$

Protein: $2x + 4y + 3z \geq 130$

Storage: $x \leq 25$

$C = 4x + 5y + 4z$

Minimise $C = 4x + 5y + 4z$ subject to $x + y + z \geq 40$, $2x + 4y + 3z \geq 130$, $x \leq 25$, $x, y, z \geq 0$

b)

$$x + y + z - s1 + a1 = 40$$

$$2x + 4y + 3z - s2 + a2 = 130$$

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$$x + s3 = 25$$

Maximise $P = -C$, penalising the artificial variables

$$P = -4x - 5y - 4z - Ma1 - Ma2$$

$$a1 + a2 = 170 - 3x - 5y - 4z + s1 + s2$$

$$P = -4x - 5y - 4z - M(170 - 3x - 5y - 4z + s1 + s2)$$

$$P + (4 - 3M)x + (5 - 5M)y + (4 - 4M)z + Ms1 + Ms2 = -170M$$

Initial tableau (columns: x, y, z, s1, s2, s3, a1, a2 | value)

a1: 1, 1, 1, −1, 0, 0, 1, 0 | 40

a2: 2, 4, 3, 0, −1, 0, 0, 1 | 130

s3: 1, 0, 0, 0, 0, 1, 0, 0 | 25

P: 4 − 3M, 5 − 5M, 4 − 4M, M, M, 0, 0, 0 | −170M

c)

Every entry in the P row is ≥ 0 (M is large)

Optimal: no negative values in the objective row

d)

Basic: $z = 30$, $y = 10$, $s3 = 25$

x, s1, s2, a1, a2 are non-basic, so zero

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$P = -170$

$C = -P = 170$

0 kg of Alfa, 10 kg of Bran, 30 kg of Corn; minimum cost £170

e)

s1 = 0 and s2 = 0: the mix is exactly 40 kg with exactly 130 units of protein

s3 = 25: none of the 25 kg of Alfa storage is used

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