Discrete random variables: expectation and variance

Worked solutions

Further Maths topic practice

This topic online

1 The discrete random variable X has the probability distribution shown.

x	1	2	3	4
$\mathbf{P}(X = x)$	0.1	0.3	k	0.2

a. Find the value of k.

b. Find E(X).

(3 marks)

a) Probabilities add up to 1

$0.1 + 0.3 + k + 0.2 = 1$

$0.6 + k = 1$

$\boxed{k = 0.4}$

b) $E(X) = \Sigma\, x\, P(X = x)$

$E(X) = 1(0.1) + 2(0.3) + 3(0.4) + 4(0.2)$

$E(X) = 0.1 + 0.6 + 1.2 + 0.8$

$\boxed{E(X) = 2.7}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

2 Priya records the number of goals, X, scored by her netball club's under-14 team in the first quarter of a match. The distribution of X is modelled as shown.

x	0	1	2	3
$P(X = x)$	0.2	0.4	0.3	0.1

Find

a. E(X)

b. E(X^2)

c. the standard deviation of X.

(4 marks)

a) $E(X) = \Sigma\, x\, P(X = x)$

$E(X) = 0(0.2) + 1(0.4) + 2(0.3) + 3(0.1)$

$E(X) = 0 + 0.4 + 0.6 + 0.3$

$\boxed{E(X) = 1.3}$

b) $E(X^2) = \Sigma\, x^2\, P(X = x)$

$E(X^2) = 0(0.2) + 1(0.4) + 4(0.3) + 9(0.1)$

$E(X^2) = 0 + 0.4 + 1.2 + 0.9$

$\boxed{E(X^2) = 2.5}$

c) $Var(X) = E(X^2) - [E(X)]^2$

$Var(X) = 2.5 - 1.3^2$

$Var(X) = 2.5 - 1.69 = 0.81$

$sd = \sqrt{0.81}$

$\boxed{0.9}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

3 The discrete random variable X has probability function

$P(X = x) = k(x + 2)$ for $x = 1, 2, 3, 4$

and $P(X = x) = 0$ otherwise, where k is a constant.

a. Show that $k = \frac{1}{18}$

b. Find $P(X > 2)$.

c. Find the exact values of $E(X)$ and $Var(X)$. (5 marks)

a) Write out the table

x: 1, 2, 3, 4

P: $3k$, $4k$, $5k$, $6k$

Probabilities add up to 1

$$3k + 4k + 5k + 6k = 1$$

$$18k = 1$$

$$\boxed{k = \frac{1}{18}}$$

b) More than 2 means 3 or 4

$$P(X > 2) = \frac{5}{18} + \frac{6}{18}$$

$$\boxed{\frac{11}{18}}$$

c) $E(X) = \Sigma\, x\, P(X = x)$

$$E(X) = \frac{(1(3) + 2(4) + 3(5) + 4(6))}{18}$$

$$E(X) = \frac{50}{18} = \frac{25}{9}$$

$$E(X^2) = \frac{(1(3) + 4(4) + 9(5) + 16(6))}{18}$$

$$E(X^2) = \frac{160}{18} = \frac{80}{9}$$

$Var(X) = E(X^2) - [E(X)]^2$

$$Var(X) = \frac{80}{9} - \left(\frac{25}{9}\right)^2$$

$$Var(X) = \frac{720}{81} - \frac{625}{81}$$

$$\boxed{E(X) = \frac{25}{9},\ Var(X) = \frac{95}{81}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

4 The discrete random variable X has the probability distribution shown, where a and b are constants.

x	−1	0	2	5
$\mathbf{P}(X = x)$	a	0.3	b	0.1

Given that $E(X) = 1.1$

a. find the value of a and the value of b

b. find $\text{Var}(X)$.

(6 marks)

a) Probabilities add up to 1

$$a + 0.3 + b + 0.1 = 1$$

$$a + b = 0.6$$

Use $E(X) = 1.1$

$$(-1)a + 0(0.3) + 2b + 5(0.1) = 1.1$$

$$2b - a = 0.6$$

Add the two equations

$$3b = 1.2$$

$$b = 0.4$$

$$a = 0.6 - 0.4$$

$$\boxed{a = 0.2,\ b = 0.4}$$

b) $E(X^2)$ first

$$E(X^2) = 1(0.2) + 0(0.3) + 4(0.4) + 25(0.1)$$

$$E(X^2) = 0.2 + 1.6 + 2.5 = 4.3$$

$$\text{Var}(X) = 4.3 - 1.1^2$$

$$\text{Var}(X) = 4.3 - 1.21$$

$$\boxed{\text{Var}(X) = 3.09}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

5 The discrete random variable X has the probability distribution shown.

x	1	2	4	5
$P(X = x)$	0.2	0.25	0.4	0.15

a. Find $E(\frac{1}{X})$.

b. Find $P(X^2 > 3X + 2)$.

(5 marks)

a) $E(g(X)) = \Sigma\, g(x)\, P(X = x)$

$$E\left(\frac{1}{X}\right) = \frac{1}{1}(0.2) + \frac{1}{2}(0.25) + \frac{1}{4}(0.4) + \frac{1}{5}(0.15)$$

$$E\left(\frac{1}{X}\right) = 0.2 + 0.125 + 0.1 + 0.03$$

0.455

b) Solve the quadratic inequality first

$$x^2 - 3x - 2 = 0$$

$$x = \frac{(3 \pm \sqrt{17})}{2}$$

$x \approx -0.56$ or $x \approx 3.56$

Positive quadratic: greater than 0 outside the roots

$x < -0.56$ or $x > 3.56$

So $x = 4$ or $x = 5$

$P(X = 4) + P(X = 5) = 0.4 + 0.15$

0.55

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

6 The discrete random variable X has the probability distribution shown.

x	−1	0	1	2
$\mathbf{P}(X = x)$	0.2	0.1	0.3	0.4

a. Find Var(X).

b. Find Var(X^2).

(6 marks)

a) $E(X)$ and $E(X^2)$

$$E(X) = (-1)(0.2) + 0(0.1) + 1(0.3) + 2(0.4)$$

$$E(X) = 0.9$$

$$E(X^2) = 1(0.2) + 0(0.1) + 1(0.3) + 4(0.4)$$

$$E(X^2) = 2.1$$

$$\text{Var}(X) = 2.1 - 0.9^2$$

$$\text{Var}(X) = 2.1 - 0.81$$

$$\boxed{\text{Var}(X) = 1.29}$$

b) $\text{Var}(X^2) = E(X^4) - [E(X^2)]^2$

$$E(X^4) = 1(0.2) + 0(0.1) + 1(0.3) + 16(0.4)$$

$$E(X^4) = 0.2 + 0.3 + 6.4 = 6.9$$

$$\text{Var}(X^2) = 6.9 - 2.1^2$$

$$\text{Var}(X^2) = 6.9 - 4.41$$

$$\boxed{\text{Var}(X^2) = 2.49}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

7 The discrete random variable X can take only the values 1, 2, 3 and 4, with

$P(X = 1) = p$, $P(X = 2) = q$, $P(X = 3) = r$, $P(X = 4) = 0.2$

Given that $E(X) = 2.5$ and $Var(X) = 1.05$, find the values of p, q and r. (7 marks)

Probabilities add up to 1

$$p + q + r + 0.2 = 1$$

$$p + q + r = 0.8 \quad (1)$$

Use $E(X) = 2.5$

$$p + 2q + 3r + 4(0.2) = 2.5$$

$$p + 2q + 3r = 1.7 \quad (2)$$

$E(X^2) = Var(X) + [E(X)]^2$

$$E(X^2) = 1.05 + 2.5^2 = 7.3$$

$$p + 4q + 9r + 16(0.2) = 7.3$$

$$p + 4q + 9r = 4.1 \quad (3)$$

(2) - (1) and (3) - (1) remove p

$$q + 2r = 0.9 \quad (4)$$

$$3q + 8r = 3.3 \quad (5)$$

(5) - 3 x (4)

$$2r = 0.6$$

$$r = 0.3$$

$$q = 0.9 - 0.6 = 0.3$$

$$p = 0.8 - 0.3 - 0.3 = 0.2$$

$$\boxed{p = 0.2,\ q = 0.3,\ r = 0.3}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

8 At a school fair, a player pays £1 to roll three fair six-sided dice. The number of sixes rolled is X. The prize, W pounds, depends on X as shown.

Number of sixes, x	0	1	2	3
Prize, w (£)	0	1	5	20

a. Show that $P(X = 2) = \frac{15}{216}$

b. Find $E(W)$.

c. The game is played 200 times. Find the profit the school should expect to make.

d. Find $Var(W)$.

(7 marks)

a) Two sixes and one non-six, in 3 possible orders

$$P(X = 2) = 3 \times (\frac{1}{6})^2 \times \frac{5}{6}$$

$$P(X = 2) = 3 \times \frac{5}{216}$$

$$\boxed{P(X = 2) = \frac{15}{216}}$$

b) The other probabilities in the same way

$$P(X = 0) = (\frac{5}{6})^3 = \frac{125}{216}$$

$$P(X = 1) = 3 \times \frac{1}{6} \times (\frac{5}{6})^2 = \frac{75}{216}$$

$$P(X = 3) = (\frac{1}{6})^3 = \frac{1}{216}$$

$E(W) = \Sigma\, w\, P(W = w)$

$$E(W) = \frac{(0(125) + 1(75) + 5(15) + 20(1))}{216}$$

$$E(W) = \frac{170}{216} = \frac{85}{108}$$

$$\boxed{£0.787}$$

c) Profit per game = £1 − expected prize

$$\text{profit per game} = 1 - \frac{85}{108} = \frac{23}{108}$$

$$200 \times \frac{23}{108} = 42.592...$$

$$\boxed{£42.59}$$

d) $Var(W) = E(W^2) - [E(W)]^2$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$$E(W^2) = \frac{(1(75) + 25(15) + 400(1))}{216}$$

$$E(W^2) = \frac{850}{216}$$

$$\text{Var}(W) = \frac{850}{216} - \left(\frac{170}{216}\right)^2$$

$$\text{Var}(W) = 3.935... - 0.619...$$

$$\text{Var}(W) = 3.3157...$$

$$\boxed{3.32}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

9 A bakery sells a type of loaf to customers. The number of these loaves, X, bought by a customer has the distribution shown.

x	1	2	3
$\mathbf{P}(X = x)$	0.5	0.3	0.2

Two customers are chosen at random, independently. They buy X_1 and X_2 loaves. Let $M = \frac{X_1 + X_2}{2}$ be the mean number of loaves they buy.

a. Find $P(X_1 + X_2 = 4)$.

b. Find $P(M \geq 2.5)$.

c. Find $E(M)$, showing your method.

(6 marks)

a) List the pairs that give a total of 4

(1, 3), (3, 1), (2, 2)

$P(\text{total} = 4) = 2(0.5)(0.2) + (0.3)^2$

$P(\text{total} = 4) = 0.2 + 0.09$

0.29

b) $M \geq 2.5$ means a total of at least 5

(2, 3), (3, 2), (3, 3)

$P(M \geq 2.5) = 2(0.3)(0.2) + (0.2)^2$

$P(M \geq 2.5) = 0.12 + 0.04$

0.16

c) Distribution of the total T

$P(T = 2) = 0.5^2 = 0.25$

$P(T = 3) = 2(0.5)(0.3) = 0.3$

$P(T = 4) = 0.29$

$P(T = 5) = 2(0.3)(0.2) = 0.12$

$P(T = 6) = 0.2^2 = 0.04$

$E(T) = 2(0.25) + 3(0.3) + 4(0.29) + 5(0.12) + 6(0.04)$

$E(T) = 0.5 + 0.9 + 1.16 + 0.6 + 0.24 = 3.4$

$E(M) = \frac{3.4}{2}$

E(M) = 1.7 (the same as E(X))

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

10 The discrete random variable X has the probability distribution shown.

x	0	1	2	3
$P(X = x)$	0.1	0.4	0.3	0.2

a. Find $E(X)$ and $Var(X)$.

The random variable Y is defined by $Y = (X - 1)^2$

b. Find the probability distribution of Y.

c. Find $E(Y)$ and verify that $E(Y) = Var(X) + (E(X) - 1)^2$

d. Find $Var(Y)$.

e. Find $P(Y > X)$.

(8 marks)

a) $E(X)$ and $E(X^2)$

$E(X) = 0 + 0.4 + 2(0.3) + 3(0.2) = 1.6$

$E(X^2) = 0 + 0.4 + 4(0.3) + 9(0.2) = 3.4$

$Var(X) = 3.4 - 1.6^2$

$\boxed{E(X) = 1.6,\ Var(X) = 0.84}$

b) Work out Y for each value of X

x: 0, 1, 2, 3

y: 1, 0, 1, 4

Combine the two values that give $y = 1$

$P(Y = 0) = 0.4$

$P(Y = 1) = 0.1 + 0.3 = 0.4$

$P(Y = 4) = 0.2$

c)

$E(Y) = 0(0.4) + 1(0.4) + 4(0.2) = 1.2$

$Var(X) + (E(X) - 1)^2 = 0.84 + 0.6^2$

$Var(X) + (E(X) - 1)^2 = 0.84 + 0.36 = 1.2$

$\boxed{E(Y) = 1.2, \text{ which matches}}$

d) $Var(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = 0(0.4) + 1(0.4) + 16(0.2) = 3.6$

$Var(Y) = 3.6 - 1.2^2$

$Var(Y) = 3.6 - 1.44$

$\boxed{2.16}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

e) Compare y with x in the table

$x = 0$: $y = 1 > 0$ yes

$x = 1$: $y = 0$ no

$x = 2$: $y = 1$ no

$x = 3$: $y = 4 > 3$ yes

$P(Y > X) = 0.1 + 0.2$

$\boxed{0.3}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

11 Mo designs a spinner for a board game. The score, X, on the spinner has probability function

$P(X = x) = \frac{k}{x}$ for $x = 1, 2, 3, 6$

and $P(X = x) = 0$ otherwise, where k is a constant.

a. Show that $k = \frac{1}{2}$

b. Show that $E(X) = Var(X)$.

c. Find the exact value of $E(\frac{1}{X})$.

(7 marks)

a) Probabilities add up to 1

$$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{6} = 1$$

$$k\left(\frac{6}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6}\right) = 1$$

$$2k = 1$$

$$\boxed{k = \frac{1}{2}}$$

b) $x \times \frac{k}{x} = k$ for every value

$$E(X) = k + k + k + k = 4k = 2$$

$x^2 \times \frac{k}{x} = kx$

$$E(X^2) = k(1 + 2 + 3 + 6) = 12k = 6$$

$$Var(X) = 6 - 2^2 = 2$$

$$\boxed{E(X) = Var(X) = 2}$$

c) $\frac{1}{x} \times \frac{k}{x} = \frac{k}{x^2}$

$$E\left(\frac{1}{X}\right) = k\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{36}\right)$$

$$E\left(\frac{1}{X}\right) = \frac{1}{2} \times \frac{(36 + 9 + 4 + 1)}{36}$$

$$E\left(\frac{1}{X}\right) = \frac{1}{2} \times \frac{50}{36}$$

$$\boxed{\frac{25}{36}}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

12 Ravi runs a bike-hire stall at a country park. The number of bikes, X, hired by a customer has probability function

$P(X = x) = a + bx$ for $x = 1, 2, 3, 4$

where a and b are constants. Given that $E(X) = 2.8$

a. find the value of a and the value of b.

Ravi gives a discount for groups, so a customer who hires x bikes is charged £C, where $C = 12X - X^2$

b. Find $E(C)$.

c. Find $Var(C)$.

d. Two customers arrive independently. Find the probability that Ravi charges them a total of at least £59

(10 marks)

a) Probabilities add up to 1

$(a + b) + (a + 2b) + (a + 3b) + (a + 4b) = 1$

$4a + 10b = 1 \quad (1)$

$E(X) = \Sigma\, x(a + bx)$

$a(1 + 2 + 3 + 4) + b(1 + 4 + 9 + 16) = 2.8$

$10a + 30b = 2.8 \quad (2)$

$(2) - 2.5 \times (1)$

$5b = 0.3$

$b = 0.06$

$4a = 1 - 0.6$

$\boxed{a = 0.1,\ b = 0.06}$

b) Table of probabilities

x: 1, 2, 3, 4

P: 0.16, 0.22, 0.28, 0.34

$E(C) = 12E(X) - E(X^2)$

$E(X^2) = 1(0.16) + 4(0.22) + 9(0.28) + 16(0.34)$

$E(X^2) = 0.16 + 0.88 + 2.52 + 5.44 = 9$

$E(C) = 12(2.8) - 9$

$\boxed{E(C) = £24.60}$

c) Charges for $x = 1, 2, 3, 4$

c: 11, 20, 27, 32

$E(C^2) = 121(0.16) + 400(0.22) + 729(0.28) + 1024(0.34)$

$E(C^2) = 19.36 + 88 + 204.12 + 348.16 = 659.64$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$\text{Var}(C) = 659.64 - 24.6^2$

$\text{Var}(C) = 659.64 - 605.16$

$\boxed{54.48}$

d) At least 59: (32, 32), (32, 27) or (27, 32)

$P = 0.34^2 + 2(0.34)(0.28)$

$P = 0.1156 + 0.1904$

$\boxed{0.306}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.