Hypothesis tests for a Poisson mean

Worked solutions

Further Maths topic practice

This topic online

1 For each situation, write down suitable null and alternative hypotheses in terms of λ, and say whether the test is one-tailed or two-tailed.

a. A garage sells used cars at a mean rate of 4 per week. After an advertising campaign the owner believes the rate has increased. λ is the mean number of cars sold per week.

b. A lake has a mean of 0.8 water lilies per square metre. A biologist believes a new fish species has reduced this. λ is the mean number of water lilies per square metre.

c. A train operator records a mean of 12 delayed trains per week on a line. After a timetable change, a passenger group thinks the mean number of delays per week has changed. They will count the delays over a **2-week** period. λ is the mean number of delays in 2 weeks.

(3 marks)

H_0 = the old mean, H_1 = what they believe

a) "increased" means bigger than 4

$H_0: \lambda = 4$

$H_1: \lambda > 4$

one-tailed

b) "reduced" means smaller than 0.8

$H_0: \lambda = 0.8$

$H_1: \lambda < 0.8$

one-tailed

c) Scale the mean to 2 weeks: $2 \times 12 = 24$

$H_0: \lambda = 24$

$H_1: \lambda \neq 24$

two-tailed ("changed" has no direction)

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2 Over many years, red kites have been seen above a hillside in Wales at a mean rate of 3 per hour. A feeding station is set up nearby. In a randomly chosen hour after it opens, 7 red kites are seen.

Test, at the 5% level of significance, whether there is evidence that the mean rate of sightings has increased. State your hypotheses clearly.

(4 marks)

λ = mean number of sightings per hour

$H_0: \lambda = 3$

$H_1: \lambda > 3$

Assume H_0 is true

$X \sim Po(3)$

p-value: probability of 7 or more

$P(X \geqslant 7) = 1 - P(X \leqslant 6)$

$P(X \geqslant 7) = 1 - 0.966491...$

$P(X \geqslant 7) = 0.0335$

Compare with 5%

$0.0335 < 0.05$, so reject H_0

Evidence that the mean rate of red kite sightings has increased

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3 A broadband provider receives reports of faults in a town at a mean rate of 4 per day. After the network is upgraded, a total of 3 faults are reported in a randomly chosen 2-day period.

Test, at the 5% level of significance, whether there is evidence that the mean rate of faults has decreased. State your hypotheses clearly.

(5 marks)

Scale the mean to 2 days: $2 \times 4 = 8$

λ = mean number of faults in 2 days

$H_0: \lambda = 8$

$H_1: \lambda < 8$

Assume H_0 is true

$X \sim Po(8)$

Lower tail: 3 or fewer

$$P(X \leq 3) = e^{-8}\left(1 + 8 + \frac{8^2}{2!} + \frac{8^3}{3!}\right)$$

$$P(X \leq 3) = e^{-8}(1 + 8 + 32 + 85.333\ldots)$$

$$P(X \leq 3) = 0.0424$$

Compare with 5%

$0.0424 < 0.05$, so reject H_0

Evidence that the mean rate of faults has decreased

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4 An online shop sells a brand of headphones at a mean rate of 5 pairs per hour. The shop starts a social media advert and wants to test whether the mean rate of sales has increased, using the number of pairs sold in one randomly chosen hour.

a. Write down suitable hypotheses.

b. Find the critical region for the test at the 5% level of significance.

c. Write down the actual significance level of the test.

(5 marks)

a) λ = mean number of pairs sold per hour

$H_0: \lambda = 5$

$H_1: \lambda > 5$

b) Under H_0, $X \sim Po(5)$

Upper tail: want $P(X \geqslant c) \leqslant 0.05$

$P(X \geqslant 9) = 1 - P(X \leqslant 8) = 0.0681$

more than 0.05

$P(X \geqslant 10) = 1 - P(X \leqslant 9) = 0.0318$

less than 0.05

Critical region $X \geqslant 10$

c) Actual significance level = $P(X$ in the critical region $\mid H_0)$

$P(X \geqslant 10) = 0.0318$

0.0318 (3.18%)

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5 The number of road accidents per year at a junction has a Poisson distribution with mean 6. The council changes the road layout and wants to test, at the 5% level of significance, whether the mean number of accidents per year has changed, using the number of accidents in the following year.

a. Find the critical region for this test. The probability in each tail should be less than 0.025 and as close to 0.025 as possible. State the probability in each tail.

b. Find the actual significance level of the test.

c. In the year after the change there are 2 accidents. State, with a reason, the conclusion of the test. (6 marks)

a) $H_0: \lambda = 6$, $H_1: \lambda \neq 6$

Two-tailed: each tail at most 0.025

$X \sim Po(6)$

Lower tail

$P(X \leqslant 1) = 7e^{-6} = 0.0174$

$P(X \leqslant 2) = 25e^{-6} = 0.0620$

0.0620 is too big, so stop at 1

Upper tail

$P(X \geqslant 11) = 1 - P(X \leqslant 10) = 0.0426$

$P(X \geqslant 12) = 1 - P(X \leqslant 11) = 0.0201$

0.0426 is too big, so start at 12

$X \leqslant 1$ (0.0174) or $X \geqslant 12$ (0.0201)

b) Add the two tail probabilities

$0.017351... + 0.020092... = 0.037443...$

0.0374

c) Is 2 in the critical region?

2 is not in the critical region, so do not reject H_0

Insufficient evidence that the mean number of accidents has changed

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6 An electrical cable has faults at a mean rate of 2.5 per 100 metres. An engineer believes a new supplier's cable has fewer faults. A randomly chosen 500-metre length of the new cable has 7 faults.

Test, at the 5% level of significance, whether there is evidence to support the engineer's belief. State your hypotheses clearly.

(5 marks)

Scale the mean to 500 m: $5 \times 2.5 = 12.5$

λ = mean number of faults per 500 m

$H_0: \lambda = 12.5$

$H_1: \lambda < 12.5$

Assume H_0 is true

$X \sim Po(12.5)$

Lower tail: 7 or fewer

$P(X \leq 7) = 0.0698$

Compare with 5%

$0.0698 > 0.05$

Not significant, so do not reject H_0

Insufficient evidence that the new cable has fewer faults

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7 A bike shop in Cambridge sells a mean of 9 bikes per week. After a new cycle lane opens, the shop sells 16 bikes in a randomly chosen week.

a. Test, at the 5% level of significance, whether there is evidence that the mean number of bikes sold per week has changed. State your hypotheses clearly.

b. State, with a reason, whether your conclusion would be the same at the 1% level of significance. (6 marks)

a) λ = mean number of bikes sold per week

$H_0: \lambda = 9$

$H_1: \lambda \neq 9$

$X \sim Po(9)$

16 is above 9, so use the upper tail

$P(X \geqslant 16) = 1 - P(X \leqslant 15)$

$P(X \geqslant 16) = 1 - 0.977964...$

$P(X \geqslant 16) = 0.0220$

Two-tailed at 5%: compare with 2.5%

$0.0220 < 0.025$, so reject H_0

Evidence that the mean number of bikes sold has changed

b) Two-tailed at 1%: compare with 0.5%

$0.0220 > 0.005$, so do not reject H_0

Not the same: at 1% there is insufficient evidence of a change

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8 A hospital ward admits patients at a mean rate of 10 per day. A new walk-in centre opens nearby and the ward manager, Siobhan, believes the mean number of admissions per day has fallen. She will test this at the 1% level of significance using the number of admissions on one randomly chosen day.

a. Find the largest number of admissions on that day that would lead Siobhan to conclude that the mean has fallen.

b. Find the actual significance level of the test.

c. On the chosen day there are 3 admissions. State the conclusion of the test in context.

(6 marks)

a) λ = mean admissions per day

$H_0: \lambda = 10$

$H_1: \lambda < 10$

$X \sim Po(10)$

Lower tail: want $P(X \leq c) \leq 0.01$

$P(X \leq 2) = e^{-10}(1 + 10 + 50) = 0.0028$

$P(X \leq 3) = e^{-10}(1 + 10 + 50 + 166.67) = 0.0103$

$0.0103 > 0.01$, so 3 is not in the critical region

2 (critical region $X \leq 2$)

b) Actual significance level

$P(X \leq 2) = 0.002769...$

0.0028

c) 3 is not in $X \leq 2$, so do not reject H_0

Insufficient evidence at 1% that the mean number of admissions has fallen

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9 Books are returned late to a village library at a mean rate of 1.5 per week. The librarian, Mr Okafor, suspects late returns are becoming more frequent. He decides to count the late returns in a 4-week period and test, at the 5% level of significance, whether the mean rate has increased.

a. Find the critical region for this test.

b. In one 4-week period there are 10 late returns. State, with a reason, whether there is evidence that the rate has increased.

Mr Okafor repeats the test for 10 separate 4-week periods. Assume that the mean rate stays at 1.5 per week throughout.

c. Find the probability that at least 2 of the 10 tests give a significant result.

(8 marks)

a) Scale the mean to 4 weeks: $4 \times 1.5 = 6$

$H_0: \lambda = 6$

$H_1: \lambda > 6$

$X \sim Po(6)$

Upper tail: want $P(X \geqslant c) \leqslant 0.05$

$P(X \geqslant 10) = 1 - P(X \leqslant 9) = 0.0839$

$P(X \geqslant 11) = 1 - P(X \leqslant 10) = 0.0426$

Critical region $X \geqslant 11$

b) 10 is not in $X \geqslant 11$

Do not reject H_0: insufficient evidence that late returns have increased

c) Each test is significant with probability 0.042621... when H_0 is true

$S \sim B(10, 0.042621...)$

$P(S = 0) = 0.957378...^{10} = 0.646903...$

$P(S = 1) = 10 \times 0.042621... \times 0.957378...^{9} = 0.287991...$

$P(S \geqslant 2) = 1 - 0.646903... - 0.287991...$

0.0651

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10 A manufacturer claims that 2% of its LED bulbs are faulty. A retailer, Grace, believes the proportion is higher. She tests a random sample of 200 bulbs and finds that 9 are faulty.

a. Explain why a Poisson distribution can be used to model the number of faulty bulbs in the sample, if the manufacturer's claim is true.

b. Using a suitable Poisson approximation, test at the 5% level of significance whether there is evidence to support Grace's belief. State your hypotheses clearly.

(7 marks)

a) Number faulty under the claim

$X \sim B(200, 0.02)$

$n = 200$ is large and $p = 0.02$ is small

$\lambda = np = 200 \times 0.02 = 4$

$\boxed{X \approx Po(4)}$

b) p = proportion of faulty bulbs

$H_0: p = 0.02$

$H_1: p > 0.02$

Under H_0, $X \approx Po(4)$

$P(X \geqslant 9) = 1 - P(X \leqslant 8)$

$P(X \geqslant 9) = 1 - 0.978637...$

$P(X \geqslant 9) = 0.0214$

Compare with 5%

$0.0214 < 0.05$, so reject H_0

$\boxed{\text{Evidence that more than 2\% of the bulbs are faulty}}$

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11 A bottling plant has two machines. Machine A breaks down at a mean rate of 1.2 times per month and machine B at a mean rate of 0.8 times per month, independently of each other.

a. Write down the distribution of the total number of breakdowns of the two machines in a 3-month period.

The plant introduces a new maintenance schedule. In the next 3 months there is a total of 1 breakdown.

b. Test, at the 5% level of significance, whether there is evidence that the mean total number of breakdowns has decreased. State your hypotheses clearly.

c. Find the critical region for this test and its actual significance level.

(8 marks)

a) Independent, so add the means

$$\text{mean per month} = 1.2 + 0.8 = 2$$

$$\text{mean for 3 months} = 3 \times 2 = 6$$

$\boxed{Po(6)}$

b) λ = mean total breakdowns in 3 months

$H_0: \lambda = 6$

$H_1: \lambda < 6$

$X \sim Po(6)$

$P(X \leq 1) = e^{-6}(1 + 6)$

$P(X \leq 1) = 7e^{-6} = 0.0174$

$0.0174 < 0.05$, so reject H_0

$\boxed{\text{Evidence that the mean number of breakdowns has decreased}}$

c) Lower tail: find the largest c with $P(X \leq c) \leq 0.05$

$P(X \leq 1) = 0.0174$

$P(X \leq 2) = e^{-6}(1 + 6 + 18) = 0.0620$

$0.0620 > 0.05$, so stop at 1

$\boxed{X \leq 1, \text{ actual significance level } 0.0174}$

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12 Boats pass through a canal lock in Birmingham at a mean rate of 2.5 per hour. After new mooring charges are introduced, the lock keeper, Ayesha, believes the mean rate has fallen. She will count the boats passing through the lock in a period of t hours and carry out a test at the 5% level of significance.

a. Show that if $t = 1$, the test can never lead Ayesha to conclude that the rate has fallen.

b. Find the smallest whole number of hours t for which the test can give a significant result.

c. For this value of t, find the critical region and the actual significance level of the test.

d. In the period she observes, 2 boats pass through the lock. State the conclusion of the test in context.

(9 marks)

a) Lower-tail test: the most extreme result is 0 boats

$t = 1$: $X \sim Po(2.5)$

$P(X = 0) = e^{-2.5} = 0.0821$

$0.0821 > 0.05$, so even 0 boats is not significant

With $t = 1$ the test can never reject H_0

b) Need $P(X = 0) \leq 0.05$ with $X \sim Po(2.5t)$

$e^{-2.5t} \leq 0.05$

$2.5t \geq \ln 20$

$t \geq 1.198...$

$t = 2$

c) $t = 2$: under H_0, $X \sim Po(5)$

$H_0: \lambda = 5$

$H_1: \lambda < 5$

$P(X \leq 1) = e^{-5}(1 + 5) = 0.0404$

$P(X \leq 2) = e^{-5}(1 + 5 + 12.5) = 0.1247$

$0.1247 > 0.05$, so stop at 1

$X \leq 1$, actual significance level 0.0404

d) 2 is not in $X \leq 1$, so do not reject H_0

Insufficient evidence that the mean rate of boats has fallen

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