# Linear transformations and the discrete uniform distribution

Worked solutions

**Further Maths topic practice**

This topic online

**1** The random variable $X$ has $E(X) = 4$ and $Var(X) = 2.5$

Find

a. $E(3X - 2)$

b. $Var(3X - 2)$

c. $Var(5 - 2X)$

(3 marks)

a) $E(aX + b) = aE(X) + b$

$E(3X - 2) = 3(4) - 2$

$\boxed{10}$

b) $Var(aX + b) = a^2 Var(X)$: adding b does not change the spread

$Var(3X - 2) = 3^2 \times 2.5$

$\boxed{22.5}$

c) $a = -2$, and $(-2)^2 = 4$

$Var(5 - 2X) = 4 \times 2.5$

$\boxed{10}$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**2** A fair twelve-sided dice has faces numbered 1 to 12. The random variable $X$ is the score when the dice is rolled once.

a. Write down the name of the distribution of $X$.

b. Find E($X$) and Var($X$).

c. Find P($X > 8$).

(4 marks)

a) Every score from 1 to 12 is equally likely

**Discrete uniform distribution**

b) $n = 12$

$$E(X) = \frac{(n+1)}{2} = \frac{13}{2}$$

$$E(X) = 6.5$$

$$Var(X) = \frac{(n^2-1)}{12} = \frac{(144-1)}{12}$$

$$Var(X) = \frac{143}{12} = 11.916...$$

$$E(X) = 6.5,\ Var(X) = \frac{143}{12}\ (11.9)$$

c) More than 8 is 9, 10, 11 or 12

$$P(X > 8) = \frac{4}{12}$$

$$\frac{1}{3}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**3** The discrete random variable $X$ has the probability distribution shown.

| $x$ | 1 | 2 | 3 |
|---|---|---|---|
| $P(X = x)$ | 0.3 | 0.5 | 0.2 |

a. Find $E(X)$ and $Var(X)$.

The random variable $Y = 2X + 3$

b. Find $E(Y)$ and $Var(Y)$.

(5 marks)

a) $E(X) = \Sigma\, x\, P(X = x)$

$E(X) = 1(0.3) + 2(0.5) + 3(0.2) = 1.9$

$E(X^2) = 1(0.3) + 4(0.5) + 9(0.2) = 4.1$

$Var(X) = 4.1 - 1.9^2$

$Var(X) = 4.1 - 3.61$

$E(X) = 1.9,\ Var(X) = 0.49$

b) $E(2X + 3) = 2E(X) + 3$

$E(Y) = 2(1.9) + 3 = 6.8$

$Var(2X + 3) = 2^2\, Var(X)$

$Var(Y) = 4 \times 0.49$

$E(Y) = 6.8,\ Var(Y) = 1.96$

**4** An orchard in Kent grades its apples by mass. The mass of an apple, $M$ grams, is coded using

$Y = \frac{M - 150}{5}$

For the apples in one crate, $E(Y) = 2.4$ and $Var(Y) = 1.44$

Find the mean and the standard deviation of the mass of an apple in this crate.

(4 marks)

Make M the subject of the code

$5Y = M - 150$

$M = 5Y + 150$

$E(aY + b) = aE(Y) + b$

$E(M) = 5(2.4) + 150$

$E(M) = 162$

$Var(aY + b) = a^2\, Var(Y)$

$Var(M) = 25 \times 1.44 = 36$

$sd = \sqrt{36} = 6$

mean 162 g, standard deviation 6 g


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**5** The random variable $X$ has a discrete uniform distribution over the integers 1, 2, 3, ..., $n$

Given that Var($X$) = 10

a. find the value of $n$

b. find E($X$)

c. find P($X \leq 4$).

(5 marks)

a) $\text{Var}(X) = \frac{(n^2 - 1)}{12}$

$\frac{(n^2 - 1)}{12} = 10$

$n^2 - 1 = 120$

$n^2 = 121$

$n = 11$ ($n$ must be positive)

b) $E(X) = \frac{(n + 1)}{2}$

$E(X) = \frac{12}{2}$

6

c) Four of the eleven equally likely values

$P(X \leq 4) = \frac{4}{11}$

$\frac{4}{11}$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**6** A fair spinner has ten equal sections labelled 3, 5, 7, 9, 11, 13, 15, 17, 19 and 21. The random variable $S$ is the number the spinner lands on.

a. By writing $S = 2X + 1$, where $X$ has a discrete uniform distribution over 1, 2, ..., 10, find E($S$) and Var($S$).

b. Find P($|S - \mu| < \sigma$), where $\mu$ and $\sigma$ are the mean and standard deviation of $S$.

(6 marks)

a) X is uniform on 1 to 10, so $n = 10$

$$E(X) = \frac{11}{2} = 5.5$$

$$Var(X) = \frac{(100 - 1)}{12} = 8.25$$

$S = 2X + 1$

$$E(S) = 2(5.5) + 1 = 12$$

$$Var(S) = 2^2 \times 8.25 = 33$$

$\boxed{E(S) = 12,\ Var(S) = 33}$

b) Find the interval $\mu \pm \sigma$

$$\sigma = \sqrt{33} = 5.744...$$

$12 - 5.744... = 6.255...$

$12 + 5.744... = 17.744...$

$6.26 < S < 17.74$ gives $S = 7, 9, 11, 13, 15, 17$

$$p = \frac{6}{10}$$

$\boxed{0.6}$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**7** The random variable $X$ has a discrete uniform distribution over the integers 1 to 9

a. Find E($X$) and show that Var($X$) = $\frac{20}{3}$

The random variable $Y = aX + b$, where $a$ and $b$ are constants and $a > 0$. Given that E($Y$) = 12 and Var($Y$) = 60

b. find the value of $a$ and the value of $b$.

(6 marks)

a) Uniform on 1 to n with n = 9

$$E(X) = \frac{(9+1)}{2} = 5$$

$$Var(X) = \frac{(81-1)}{12}$$

$$Var(X) = \frac{80}{12} = \frac{20}{3}$$

$$\boxed{E(X) = 5,\ Var(X) = \frac{20}{3}}$$

b) Use the variance first, because b does not affect it

$$a^2 \times \frac{20}{3} = 60$$

$$a^2 = 9$$

$$a = 3 \ (a > 0)$$

$E(Y) = aE(X) + b$

$$3(5) + b = 12$$

$$\boxed{a = 3,\ b = -3}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**8** At a charity stall in Leeds, a player pays £1 and rolls a fair six-sided dice once. If the dice shows $X$, the player wins $20X$ pence.

a. Find the expected amount the player wins.

b. Find the expected profit the stall makes on each game.

c. Find the standard deviation of the amount the player wins.

d. The charge for a game is changed to $c$ pence. Find the smallest whole number $c$ for which the expected profit on a game is at least 40% of the charge.

(7 marks)

a) X is discrete uniform on 1 to 6

$$E(X) = \frac{7}{2} = 3.5$$

$$E(20X) = 20 \times 3.5$$

**70p**

b) Profit = charge − expected winnings

$$\text{profit} = 100 - 70$$

**30p**

c) $\text{Var}(X) = \frac{(n^2 - 1)}{12}$

$$\text{Var}(X) = \frac{35}{12}$$

$$\text{Var}(20X) = 400 \times \frac{35}{12} = 1166.66...$$

$$\text{sd} = \sqrt{1166.66...} = 34.15...$$

**34.2p**

d) Expected profit ≥ 40% of c

$c - 70 \geq 0.4c$

$0.6c \geq 70$

$c \geq 116.66...$

**$c = 117$**


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**9** Raffle tickets are numbered 1, 2, 3, ..., $n$. One ticket is drawn at random and its number is $X$. Given that $P(X \leq 5) = 0.25$

a. find the value of $n$

b. find $E(X)$ and $Var(X)$.

The random variable $Y = 4X - 2$

c. Find $E(Y)$ and $Var(Y)$.

(6 marks)

a) Each number has probability $\frac{1}{n}$

$$\frac{5}{n} = 0.25$$

$$\boxed{n = 20}$$

b) Uniform on 1 to 20

$$E(X) = \frac{21}{2} = 10.5$$

$$Var(X) = \frac{(400 - 1)}{12} = \frac{399}{12}$$

$$\boxed{E(X) = 10.5,\ Var(X) = 33.25}$$

c) $E(4X - 2) = 4E(X) - 2$

$$E(Y) = 4(10.5) - 2 = 40$$

$Var(4X - 2) = 16\ Var(X)$

$$Var(Y) = 16 \times 33.25 = 532$$

$$\boxed{E(Y) = 40,\ Var(Y) = 532}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**10** The discrete random variable $X$ has the probability distribution shown, where $a$ and $b$ are constants.

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(X = x)$ | 0.1 | $a$ | $b$ | 0.3 |

Given that $E(2X + 1) = 5$

a. find the value of $a$ and the value of $b$

b. find $Var(5 - 3X)$

c. find $E(5 - 3X)$.

(7 marks)

a) Undo the linear function to get E(X)

$$2E(X) + 1 = 5$$

$$E(X) = 2$$

Probabilities add to 1

$$0.1 + a + b + 0.3 = 1$$

$$a + b = 0.6 \quad (1)$$

$$0(0.1) + a + 2b + 3(0.3) = 2$$

$$a + 2b = 1.1 \quad (2)$$

(2) - (1)

$$b = 0.5$$

$$\boxed{a = 0.1,\ b = 0.5}$$

b) Var(X) first

$$E(X^2) = 0 + 1(0.1) + 4(0.5) + 9(0.3) = 4.8$$

$$Var(X) = 4.8 - 2^2 = 0.8$$

$Var(5 - 3X) = (-3)^2\ Var(X)$

$$Var(5 - 3X) = 9 \times 0.8$$

$$\boxed{7.2}$$

c) $E(5 - 3X) = 5 - 3E(X)$

$$E(5 - 3X) = 5 - 3(2)$$

$$\boxed{E(5 - 3X) = -1}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**11** The random variable $X$ has a discrete uniform distribution over the integers 1, 2, ..., $n$

a. Using $\Sigma r = \frac{1}{2}n(n+1)$ and $\Sigma r^2 = \frac{1}{6}n(n+1)(2n+1)$, prove that $E(X) = \frac{n+1}{2}$ and $Var(X) = \frac{n^2-1}{12}$

A school's sports day raffle sells tickets numbered 11 to 30 inclusive. One ticket is drawn at random and its number is $T$.

b. Find $E(T)$ and $Var(T)$.

c. Find the probability that the number on the ticket is a multiple of 4

(9 marks)

a) Each value has probability $\frac{1}{n}$

$$E(X) = \sum r \times \frac{1}{n} = \frac{1}{n}\sum r$$

$$E(X) = \frac{1}{n} \times \frac{n(n+1)}{2}$$

$$E(X) = \frac{(n+1)}{2}$$

$$E(X^2) = \frac{1}{n}\sum r^2 = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6}$$

$$E(X^2) = \frac{(n+1)(2n+1)}{6}$$

$Var(X) = E(X^2) - [E(X)]^2$

$$Var(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

Common denominator 12, take out the factor $(n+1)$

$$Var(X) = \frac{(n+1)[2(2n+1) - 3(n+1)]}{12}$$

$$Var(X) = \frac{(n+1)(4n+2-3n-3)}{12}$$

$$Var(X) = \frac{(n+1)(n-1)}{12}$$

$$\boxed{Var(X) = \frac{(n^2-1)}{12}}$$

b) $T = X + 10$ where $X$ is uniform on 1 to 20

$$E(T) = \frac{21}{2} + 10 = 20.5$$

Adding 10 does not change the variance

$$Var(T) = \frac{(20^2-1)}{12} = \frac{399}{12}$$

$$\boxed{E(T) = 20.5,\ Var(T) = 33.25}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

c) List the multiples of 4 from 11 to 30

12, 16, 20, 24, 28

$$p = \frac{5}{20}$$

$$\boxed{\frac{1}{4}}$$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

**12** A fair spinner is numbered 1, 2, 3, ..., $n$. The number it lands on is $X$. In a game at a school fete, a player's score is $S = 3X - 2$

a. Given that E($S$) = 13, show that $n = 9$

b. Find Var($S$).

A player pays 75p for one spin and receives 5$S$ pence.

c. Find the profit the fete should expect to make from 300 spins.

d. Find the standard deviation of the amount a player receives.

e. Find the probability that a player receives more than they paid.

(9 marks)

a) $E(S) = 3E(X) - 2$

$3E(X) - 2 = 13$

$E(X) = 5$

Discrete uniform: $E(X) = \frac{(n+1)}{2}$

$\frac{(n+1)}{2} = 5$

$\boxed{n = 9}$

b) $Var(X) = \frac{(n^2 - 1)}{12}$

$Var(X) = \frac{80}{12} = \frac{20}{3}$

$Var(S) = 3^2 \times \frac{20}{3}$

$\boxed{60}$

c) Expected amount received = 5E(S)

$E(5S) = 5 \times 13 = 65p$

profit per spin = $75 - 65 = 10p$

$300 \times 10p = 3000p$

$\boxed{£30}$

d) $Var(5S) = 25\ Var(S)$

$Var(5S) = 25 \times 60 = 1500$

$sd = \sqrt{1500} = 38.72...$

$\boxed{38.7p}$

e) Receives more than 75p

$5S > 75$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

$S > 15$

$3X - 2 > 15$

$X > \frac{17}{3}$, so $X = 6, 7, 8$ or $9$

$p = \frac{4}{9}$

$\boxed{\frac{4}{9}}$


Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.