The Poisson distribution

Worked solutions

Further Maths topic practice

This topic online

1 The random variable $X \sim \text{Po}(3.5)$

a. Find $P(X = 2)$.

b. Find $P(X \geq 2)$.

c. Write down the standard deviation of X.

(4 marks)

a) $P(X = r) = \frac{(e^{-\lambda} \lambda^r)}{r!}$

$P(X = 2) = \frac{(e^{-3.5} \times 3.5^2)}{2!}$

$P(X = 2) = 0.18495...$

0.1850

b) Use $1 - P(X \leq 1)$

$P(X = 0) = e^{-3.5} = 0.030197...$

$P(X = 1) = 3.5e^{-3.5} = 0.105691...$

$P(X \geq 2) = 1 - (0.030197... + 0.105691...)$

$P(X \geq 2) = 1 - 0.135888...$

0.8641

c) Mean = variance = λ

$sd = \sqrt{3.5} = 1.870...$

1.87

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

2 Emails arrive in the inbox of a dental surgery in Cardiff at a mean rate of 6 per hour.

a. State two conditions needed for a Poisson distribution to be a suitable model for the number of emails that arrive in an hour.

b. Find the probability that exactly 4 emails arrive in a randomly chosen hour.

c. Find the probability that more than 2 emails arrive in a randomly chosen 20-minute period. (5 marks)

a) Conditions in context

Emails arrive independently of each other

Emails arrive singly (one at a time)

Emails arrive at a constant average rate

b) X = emails in an hour, $X \sim Po(6)$

$$P(X = 4) = \frac{(e^{-6} \times 6^4)}{4!}$$

$$P(X = 4) = 0.13385...$$

$$\boxed{0.1339}$$

c) Scale the mean to 20 minutes

$$\lambda = 6 \times \frac{20}{60} = 2$$

$Y \sim Po(2)$

$$P(Y \leq 2) = e^{-2}\left(1 + 2 + \frac{2^2}{2!}\right)$$

$$P(Y \leq 2) = 5e^{-2} = 0.676676...$$

$$P(Y > 2) = 1 - 0.676676...$$

$$\boxed{0.3233}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

3 Kofi checks 80 pages of a draft textbook and records the number of typing errors on each page.

Number of errors	0	1	2	3	4	5
Number of pages	13	24	22	13	6	2

a. Find the mean and the variance of the number of errors per page.

b. Explain how your answers to part (a) support the use of a Poisson distribution to model the number of errors on a page.

c. Using a Poisson model with the mean from part (a), find the probability that a page has no errors. (5 marks)

a) Totals from the table

$$\Sigma f = 80$$

$$\Sigma fx = 0 + 24 + 44 + 39 + 24 + 10 = 141$$

$$\Sigma fx^2 = 0 + 24 + 88 + 117 + 96 + 50 = 375$$

$$\text{mean} = \frac{141}{80} = 1.7625$$

$$\text{variance} = \frac{375}{80} - 1.7625^2$$

$$\text{variance} = 4.6875 - 3.10640... = 1.58109...$$

mean 1.76, variance 1.58

b) Poisson: mean = variance

1.76 and 1.58 are close, so a Poisson model is reasonable

c) $X \sim Po(1.7625)$

$$P(X = 0) = e^{-1.7625}$$

$$P(X = 0) = 0.17161...$$

0.1716

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

4 Cars pass a school gate at random at a mean rate of 1.5 per minute.

a. Find the probability that no cars pass the gate in a 2-minute period.

b. Find the probability that at least 8 cars pass the gate in a 5-minute period.

(5 marks)

a) Mean for 2 minutes = 2 × 1.5

$X \sim Po(3)$

$P(X = 0) = e^{-3}$

$P(X = 0) = 0.049787...$

0.0498

b) Mean for 5 minutes = 5 × 1.5

$Y \sim Po(7.5)$

At least 8 = 1 - P(at most 7)

$P(Y \geqslant 8) = 1 - P(Y \leqslant 7)$

$P(Y \geqslant 8) = 1 - 0.524638...$

0.4754

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

5 In a 10-minute period, the number of robins visiting a garden bird feeder is modelled by $R \sim \text{Po}(2.4)$ and the number of blackbirds visiting the feeder is modelled by $B \sim \text{Po}(1.6)$

a. State an assumption needed for $R + B$ to have a Poisson distribution.

b. Find the probability that exactly 5 of these birds visit the feeder in a 10-minute period.

c. Find the probability that at least 3 of these birds visit the feeder in a 10-minute period.

(5 marks)

a)

The numbers of robins and blackbirds visiting are independent

b) Add the means: $T = R + B$

$T \sim Po(2.4 + 1.6) = Po(4)$

$$P(T = 5) = \frac{(e^{-4} \times 4^5)}{5!}$$

$P(T = 5) = 0.156293...$

0.1563

c) At least 3 = 1 - P(at most 2)

$$P(T \leq 2) = e^{-4}(1 + 4 + \frac{4^2}{2!})$$

$P(T \leq 2) = 13e^{-4} = 0.238103...$

$P(T \geq 3) = 1 - 0.238103...$

0.7619

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

6 The random variable $X \sim \text{Po}(\lambda)$, where $\lambda > 0$. Given that

$\text{P}(X = 3) = 2 \times \text{P}(X = 2)$

a. show that $\lambda = 6$

b. find $\text{P}(X \geq 5)$.

(5 marks)

a) Write both probabilities with the formula

$$\frac{(e^{-\lambda}\lambda^3)}{3!} = 2 \times \frac{(e^{-\lambda}\lambda^2)}{2!}$$

$$\frac{(e^{-\lambda}\lambda^3)}{6} = e^{-\lambda}\lambda^2$$

Divide by $e^{-\lambda}\lambda^2$, which is not 0

$$\frac{\lambda}{6} = 1$$

$$\boxed{\lambda = 6}$$

b) At least 5 = 1 - P(at most 4)

$$P(X \leq 4) = e^{-6}(1 + 6 + 18 + 36 + 54)$$

$$P(X \leq 4) = 115e^{-6} = 0.285057...$$

$$P(X \geq 5) = 1 - 0.285057...$$

$$\boxed{0.7149}$$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

7 Calls to a council helpline arrive at random at a mean rate of 1.2 every 10 minutes.

a. Find the probability that no calls arrive in a randomly chosen 10-minute period.

A 2-hour shift is split into twelve 10-minute periods.

b. Find the probability that exactly 3 of these periods have no calls.

c. Find the probability that more than 15 calls arrive during the 2-hour shift. (6 marks)

a) $X \sim Po(1.2)$

$P(X = 0) = e^{-1.2} = 0.301194...$

0.3012

b) Each period has no calls with probability $p = 0.301194...$

$Y \sim B(12, 0.301194...)$

$P(Y = 3) = 12C3 \times 0.301194...^{3} \times 0.698806...^{9}$

$P(Y = 3) = 220 \times 0.027323... \times 0.039738...$

$P(Y = 3) = 0.23887...$

0.2389

c) Mean for 2 hours = 12×1.2

$C \sim Po(14.4)$

$P(C > 15) = 1 - P(C \leq 15)$

$P(C > 15) = 1 - 0.629301...$

0.3707

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

8 A seed company states that 1.2% of its sunflower seeds fail to germinate. Amira sows 250 of these seeds. The number that fail to germinate is X.

a. Write down the exact distribution of X.

b. Explain why a Poisson distribution can be used to approximate the distribution of X.

c. Use a Poisson approximation to find $P(X \leq 2)$.

The exact value of $P(X \leq 2)$ is 0.4218 to 4 decimal places.

d. Comment on the accuracy of your approximation.

(6 marks)

a) Fixed number of independent seeds, same probability

$X \sim B(250, 0.012)$

b)

$n = 250$ is large and $p = 0.012$ is small

c) Mean $\lambda = np$

$\lambda = 250 \times 0.012 = 3$

$X \approx Po(3)$

$P(X \leq 2) = e^{-3}(1 + 3 + \frac{3^2}{2!})$

$P(X \leq 2) = 8.5e^{-3} = 0.423190...$

0.4232

d) Compare with the exact value

$0.4232 - 0.4218 = 0.0014$

Very close, so the Poisson approximation is good here

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

9 The number of items handed in to the lost property office at a railway station each day is modelled by a Poisson distribution with mean 2.5

a. Find the probability that at least 2 items are handed in on a randomly chosen day.

b. On a particular day at least 2 items are handed in. Find the probability that exactly 4 items are handed in on that day.

(6 marks)

a) $X \sim Po(2.5)$

$P(X \leq 1) = e^{-2.5}(1 + 2.5) = 0.287297...$

$P(X \geq 2) = 1 - 0.287297...$

$P(X \geq 2) = 0.712703...$

$\boxed{0.7127}$

b) $P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$, and $X = 4$ is inside $X \geq 2$

$$P(X = 4) = \frac{(e^{-2.5} \times 2.5^4)}{4!} = 0.133602...$$

$$P(X = 4 \mid X \geq 2) = \frac{0.133602...}{0.712703...}$$

$P(X = 4 \mid X \geq 2) = 0.18745...$

$\boxed{0.1875}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

10 Bird ringers at a nature reserve in Norfolk catch a rare warbler at random at a mean rate of 0.8 per hour.

a. Find the probability that they catch exactly 2 rare warblers in a 3-hour session.

b. The ringers want the probability of catching at least one rare warbler in a session to be more than 0.95. Find the smallest whole number of hours a session should last.

(6 marks)

a) Mean for 3 hours = 3×0.8

$X \sim Po(2.4)$

$$P(X = 2) = \frac{(e^{-2.4} \times 2.4^2)}{2!}$$

$P(X = 2) = 0.261268...$

0.2613

b) In t hours, $Y \sim Po(0.8t)$

$P(Y \geq 1) = 1 - e^{-0.8t}$

$1 - e^{-0.8t} > 0.95$

$e^{-0.8t} < 0.05$

Take ln of both sides

$-0.8t < \ln 0.05$

Multiply by -1 (reverse the inequality)

$0.8t > -\ln 0.05$

$0.8t > \ln 20$

$t > 3.744...$

Check the whole numbers either side

$t = 3: 1 - e^{-2.4} = 0.9093$

$t = 4: 1 - e^{-3.2} = 0.9592$

4 hours

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

11 A factory makes phone screens. Each screen, independently, is faulty with probability p. The probability that at least one screen in a random sample of 4 screens is faulty is 0.0394 to 4 decimal places.

a. Show that $p = 0.01$ to 2 decimal places.

The factory makes 600 screens each day. Use $p = 0.01$

b. Find the mean and variance of the number of faulty screens made in a day.

c. Use a suitable approximation to find the probability that more than 8 faulty screens are made in a day.

d. Explain why your answers to part (b) support the use of the approximation in part (c).

(9 marks)

a) At least one = 1 − P(none)

$1 - (1 - p)^4 = 0.0394$

$(1 - p)^4 = 0.9606$

$1 - p = 0.9606^{\frac{1}{4}} = 0.990001...$

$p = 0.009998...$

$p = 0.01$ (2 d.p.)

b) $F \sim B(600, 0.01)$

mean $= np = 600 \times 0.01 = 6$

variance $= np(1 - p) = 6 \times 0.99 = 5.94$

mean 6, variance 5.94

c) n large, p small: $F \approx Po(6)$

$P(F > 8) = 1 - P(F \leq 8)$

$P(F > 8) = 1 - 0.847237...$

0.1528

d) For a Poisson, mean = variance

6 and 5.94 are almost equal, so a Poisson model fits well

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

12 A hardware shop in Bristol sells kettles at random at a mean rate of 2.5 per day and toasters at random at a mean rate of 1.5 per day. Sales of kettles and toasters are independent.

a. Find the probability that the shop sells a total of exactly 6 kettles and toasters on a randomly chosen day.

b. Find the probability that the shop sells more than 25 kettles and toasters in a 5-day week.

c. On one day the shop sells a total of 5 kettles and toasters. Find the probability that exactly 2 of them are kettles.

d. Find the probability that, in 4 randomly chosen 5-day weeks, there is at least one week in which more than 25 kettles and toasters are sold.

(10 marks)

a) Add the means for one day

$T \sim Po(2.5 + 1.5) = Po(4)$

$P(T = 6) = \frac{(e^{-4} \times 4^6)}{6!}$

$P(T = 6) = 0.104196...$

$\boxed{0.1042}$

b) Mean for 5 days = 5 x 4

$W \sim Po(20)$

$P(W > 25) = 1 - P(W \leq 25)$

$P(W > 25) = 1 - 0.887815...$

$P(W > 25) = 0.112185...$

$\boxed{0.1122}$

c) 2 kettles and 3 toasters, given a total of 5

$P(K = 2) = \frac{(e^{-2.5} \times 2.5^2)}{2!} = 0.256516...$

$P(L = 3) = \frac{(e^{-1.5} \times 1.5^3)}{3!} = 0.125511...$

$P(T = 5) = \frac{(e^{-4} \times 4^5)}{5!} = 0.156293...$

$P(K = 2 \mid T = 5) = \frac{(0.256516... \times 0.125511...)}{0.156293...}$

$P(K = 2 \mid T = 5) = 0.20599...$

$\boxed{0.2060}$

d) Independent weeks: 1 - P(no such week)

$P = 1 - (1 - 0.112185...)^4$

$P = 1 - 0.621284...$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

0.3787

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.