Hypothesis tests for the geometric distribution

Worked solutions

Further Maths topic practice

This topic online

1 In each situation X is the number of trials up to and including the first success. Write down suitable null and alternative hypotheses in terms of p, the probability of a success, and state whether small or large values of X would support the alternative hypothesis.

a. Jack thinks his dice shows a six less often than a fair dice would. He rolls it until he gets a six.

b. An office receives spam emails, and 5% of them used to get past the spam filter. After an update, the IT manager believes this proportion has increased. She counts the spam emails until one gets past the filter.

c. A machine used to jam on 30% of its runs. After a service, an engineer wants to test whether this proportion has changed. He counts the runs until the machine jams.

(3 marks)

Smaller p means a longer wait, so large X; larger p means small X

a) "less often" means p is smaller

$H_0: p = \frac{1}{6}$

$H_1: p < \frac{1}{6}$

Large values of X

b) "increased" means p is bigger

$H_0: p = 0.05$

$H_1: p > 0.05$

Small values of X

c) "changed" has no direction

$H_0: p = 0.3$

$H_1: p \neq 0.3$

Very small or very large values of X (two-tailed)

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

2 A puzzle app claims that 25% of its daily puzzles are rated 'easy'. Daniel thinks that the proportion is lower than this. He opens puzzles one at a time and the first easy puzzle is the 12th one he opens.

Test, at the 5% level of significance, whether there is evidence to support Daniel's belief. State your hypotheses clearly.

(4 marks)

p = proportion of puzzles rated easy

$H_0: p = 0.25$

$H_1: p < 0.25$

Assume H_0 is true

$X \sim Geo(0.25)$

Large X supports H_1: find $P(12 \text{ or more})$

$P(X \geqslant 12) = 0.75^{11}$

$P(X \geqslant 12) = 0.0422$

Compare with 5%

$0.0422 < 0.05$, so reject H_0

Evidence that fewer than 25% of puzzles are rated easy

3 Of a certain type of houseplant, 5% grow with variegated leaves. Cerys believes that a new compost increases this proportion. She examines plants grown in the new compost one at a time, and the first plant with variegated leaves is the 3rd plant she examines.

Test, at the 10% level of significance, whether there is evidence to support Cerys's belief. State your hypotheses clearly.

(4 marks)

p = proportion of plants with variegated leaves

$H_0: p = 0.05$

$H_1: p > 0.05$

Assume H_0 is true

$X \sim Geo(0.05)$

Small X supports H_1: find $P(3 \text{ or fewer})$

$P(X \leqslant 3) = 1 - 0.95^3$

$P(X \leqslant 3) = 0.1426$

Compare with 10%

$0.1426 > 0.10$, so do not reject H_0

Not enough evidence that the new compost increases the proportion

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

4 At a café, 15% of customers usually buy a pastry. After a price rise, the manager wants to test whether this proportion has fallen. She will count the customers, one at a time, until the first customer who buys a pastry.

a. Write down suitable hypotheses for the test.

b. Find the critical region for the test at the 5% level of significance.

c. Find the actual significance level of the test.

(5 marks)

a) p = proportion of customers buying a pastry

$H_0: p = 0.15$

$H_1: p < 0.15$

b) Under H_0, $X \sim Geo(0.15)$; large X supports H_1

$P(X \geqslant c) = (1-p)^{c-1}$, want $\leqslant 0.05$

$0.85^{c-1} \leqslant 0.05$

$(c-1)\ln 0.85 \leqslant \ln 0.05$

$\ln 0.85$ is negative, so the inequality flips

$$c - 1 \geqslant \frac{\ln 0.05}{\ln 0.85} = 18.43...$$

$c = 20$

Check either side

$P(X \geqslant 19) = 0.85^{18} = 0.0536$

$P(X \geqslant 20) = 0.85^{19} = 0.0456$

Critical region $X \geqslant 20$

c) Actual significance level = $P(X \text{ in critical region} \mid H_0)$

0.0456

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

5 A courier company finds that 1% of the parcels it delivers are damaged. After moving to a new depot, a manager suspects that this proportion has increased. To test this, parcels from the new depot are inspected one at a time until a damaged parcel is found.

a. Find the critical region for a test at the 5% level of significance, stating your hypotheses clearly.

b. Find the actual significance level of the test.

(5 marks)

a) p = proportion of damaged parcels

$H_0: p = 0.01$

$H_1: p > 0.01$

Under H_0, $X \sim Geo(0.01)$; small X supports H_1

$P(X \leqslant c) = 1 - (1-p)^c$, want $\leqslant 0.05$

$1 - 0.99^c \leqslant 0.05$

$0.99^c \geqslant 0.95$

$c \ln 0.99 \geqslant \ln 0.95$

$\ln 0.99$ is negative, so the inequality flips

$$c \leqslant \frac{\ln 0.95}{\ln 0.99} = 5.10\ldots$$

Check either side

$P(X \leqslant 5) = 1 - 0.99^5 = 0.0490$

$P(X \leqslant 6) = 1 - 0.99^6 = 0.0585$

Critical region $X \leqslant 5$

b) Actual significance level

0.0490

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

6 On a website, 4% of visitors click on a banner advert. The page layout is changed and the web designer wants to test whether the proportion of visitors who click on the advert has changed. She will record the number of visitors, X, up to and including the first visitor who clicks on the advert.

a. Using a 10% level of significance, find the critical region for this test. The probability in each tail should be as close as possible to, but not more than, 5%

b. Find the actual significance level of the test.

(6 marks)

$H_0: p = 0.04$

$H_1: p \neq 0.04$

Under H_0, $X \sim Geo(0.04)$

a) Lower tail: want $P(X \leqslant c) \leqslant 0.05$

$P(X \leqslant 1) = 0.04$

$P(X \leqslant 2) = 1 - 0.96^2 = 0.0784$

Lower tail is $X = 1$

Upper tail: want $P(X \geqslant c) = 0.96^{c-1} \leqslant 0.05$

$$c - 1 \geqslant \frac{\ln 0.05}{\ln 0.96} = 73.39...$$

$c = 75$

$P(X \geqslant 74) = 0.96^{73} = 0.0508$

$P(X \geqslant 75) = 0.96^{74} = 0.0488$

Critical region $X = 1$ or $X \geqslant 75$

b) Add the two tails

Actual significance level $= 0.04 + 0.0488$

0.0888

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

7 A coin-pusher machine at an amusement arcade is set so that 25% of plays give a payout. Ravi plays until he gets a payout, and records the number of plays, X.

a. Ravi wants to test, at the 5% level of significance, whether the probability of a payout is greater than 0.25. Explain why this test can never lead to H_0 being rejected.

b. Find the smallest significance level at which a result of $X = 1$ would lead to H_0 being rejected in this test.

c. Ravi's friend Asha instead tests, at the 5% level, whether the probability of a payout is less than 0.25. Find the critical region for Asha's test and its actual significance level.

(6 marks)

Under H_0, $X \sim Geo(0.25)$

a) H_1: $p > 0.25$ means small X, critical region $X \leqslant c$

$P(X \leqslant 1) = 0.25$

$0.25 > 0.05$ even for the smallest value

No value of X can be in the critical region, so H_0 is never rejected

b) Need the level to be at least $P(X \leqslant 1)$

25%

c) H_1: $p < 0.25$ means large X, want $0.75^{c-1} \leqslant 0.05$

$$c - 1 \geqslant \frac{\ln 0.05}{\ln 0.75} = 10.41...$$

$c = 12$

$P(X \geqslant 11) = 0.75^{10} = 0.0563$

$P(X \geqslant 12) = 0.75^{11} = 0.0422$

Critical region $X \geqslant 12$, actual significance level 0.0422

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

8 A spinner is designed so that it lands on red with probability $\frac{1}{5}$ Ines believes that it lands on red less often than this. She will spin it until it lands on red, and record the number of spins, X.

a. Find the critical region for a test of Ines's belief at the 1% level of significance, stating your hypotheses clearly.

b. Find the actual significance level of the test.

The spinner lands on red for the first time on the 19th spin.

c. State, with a reason, the conclusion of the test at the 1% level.

d. Determine whether the conclusion would be different at the 5% level.

(7 marks)

a) p = P(Lands on red)

$H_0: p = 0.2$

$H_1: p < 0.2$

Under H_0, $X \sim Geo(0.2)$; Large X supports H_1

$0.8^{c-1} \leq 0.01$

$c - 1 \geqslant \frac{\ln 0.01}{\ln 0.8} = 20.64...$

$c = 22$

$P(X \geqslant 21) = 0.8^{20} = 0.0115$

$P(X \geqslant 22) = 0.8^{21} = 0.00922$

Critical region $X \geqslant 22$

b) Actual significance level

0.00922

c) Is 19 in the critical region?

$19 < 22$, so do not reject H_0

No evidence at 1% that the spinner lands on red less often

d) Probability of 19 or more under H_0

$P(X \geqslant 19) = 0.8^{18} = 0.0180$

$0.0180 < 0.05$, so reject H_0 at 5%

Yes: at 5% there is evidence it lands on red less often

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

9 A toy company claims that 30% of its mystery boxes contain a rare figure. Tamsin and Harvey both want to test, at the 5% level of significance, whether the proportion is less than 30%.

- Tamsin will open 15 boxes and record the number, Y, that contain a rare figure.
- Harvey will open boxes one at a time until he finds a rare figure, and record the number of boxes, X, that he opens.

a. Find the critical region for Tamsin's test and its actual significance level.

b. Find the critical region for Harvey's test and its actual significance level.

c. Under the company's claim, find the expected number of boxes Harvey opens, and hence give one advantage of Harvey's test compared with Tamsin's.

(8 marks)

$H_0: p = 0.3$

$H_1: p < 0.3$

a) Tamsin: $Y \sim B(15, 0.3)$, small Y supports H_1

$P(Y = 0) = 0.7^{15} = 0.00475$

$P(Y \leqslant 1) = 0.00475 + 15 \times 0.3 \times 0.7^{14} = 0.0353$

$P(Y \leqslant 2) = 0.1268$

$Y \leqslant 1$, actual significance level 0.0353

b) Harvey: $X \sim Geo(0.3)$, large X supports H_1

$0.7^{c-1} \leqslant 0.05$

$c - 1 \geqslant \dfrac{\ln 0.05}{\ln 0.7} = 8.40...$

$c = 10$

$P(X \geqslant 9) = 0.7^8 = 0.0576$

$P(X \geqslant 10) = 0.7^9 = 0.0404$

$X \geqslant 10$, actual significance level 0.0404

c) Expected number of boxes under H_0

$E(X) = \dfrac{1}{0.3} = 3.33$

Far fewer than Tamsin's 15 boxes on average

and 0.0404 is closer to 5% than 0.0353

3.33 boxes; Harvey's test needs fewer boxes on average

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

10 In archery practice, each of Seren's arrows hits the gold with probability 0.3, independently. A round consists of 3 arrows, and a round is called **good** if at least 2 of the 3 arrows hit the gold.

a. Show that the probability that a round is good is 0.216

Seren changes to a new bow. Her coach suspects that the probability of a good round has decreased. Seren shoots rounds until she has a good round, and the coach records the number of rounds, R.

b. Find the critical region for a test of the coach's suspicion at the 5% level of significance, stating your hypotheses clearly.

c. Write down the actual significance level of the test.

d. Seren's first good round with the new bow is her 11th round. State the conclusion of the test in context.

(8 marks)

a) G = golds in a round, $G \sim B(3, 0.3)$

$P(G \geqslant 2) = {}^3C_2 \times 0.3^2 \times 0.7 + 0.3^3$

$P(G \geqslant 2) = 0.189 + 0.027$

$\boxed{P(\text{good round}) = 0.216}$

b) $p = P(\text{good round})$

$H_0: p = 0.216$

$H_1: p < 0.216$

Under H_0, $R \sim \text{Geo}(0.216)$; Large R supports H_1

$0.784^{c-1} \leqslant 0.05$

$c - 1 \geqslant \dfrac{\ln 0.05}{\ln 0.784} = 12.31\ldots$

$c = 14$

$P(R \geqslant 13) = 0.784^{12} = 0.0539$

$P(R \geqslant 14) = 0.784^{13} = 0.0423$

$\boxed{\text{Critical region } R \geqslant 14}$

c) Actual significance level

$\boxed{0.0423}$

d) Is 11 in the critical region?

$11 < 14$, so do not reject H_0

$\boxed{\begin{array}{l}\text{No evidence that the probability of a good round has}\\ \text{decreased}\end{array}}$

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

11 Each of 25 students in a class is given a dice and asked to test whether it shows a six less often than a fair dice would. Each student rolls their dice until it shows a six, records the number of rolls, X, and carries out a test at the 5% level of significance.

a. Find the critical region for each student's test, stating the hypotheses.

b. Find the actual significance level of the test.

c. Given that all 25 dice are in fact fair, find the probability that at least 2 of the students reject H_0.

d. One student's first six comes on roll 20, and she says her dice is biased. Comment on her claim in the light of your answer to part (c).

(9 marks)

a) $p = P(\text{six})$

$H_0: p = \frac{1}{6}$

$H_1: p < \frac{1}{6}$

Under H_0, $X \sim \text{Geo}(\frac{1}{6})$; Large X supports H_1

$(\frac{5}{6})^{c-1} \leq 0.05$

$c - 1 \geqslant \frac{\ln 0.05}{\ln(5/6)} = 16.43...$

$c = 18$

$P(X \geqslant 17) = (\frac{5}{6})^{16} = 0.0541$

$P(X \geqslant 18) = (\frac{5}{6})^{17} = 0.0451$

Critical region $X \geqslant 18$

b) Actual significance level

0.0451

c) N = number of students who reject H_0

$N \sim B(25, 0.045073)$

$P(N = 0) = 0.954927^{25} = 0.3157$

$P(N = 1) = 25 \times 0.045073 \times 0.954927^{24} = 0.3725$

$P(N \geqslant 2) = 1 - 0.3157 - 0.3725$

0.3118

d) $20 \geqslant 18$, so her result is significant

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

But with 25 fair dice, P(at least one rejects) = 1 − 0.3157 = 0.68

so a significant result is quite likely just by chance

Weak evidence that her dice is biased: she should test it again

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

12 The number of faults per day on a factory's packing line has a Poisson distribution with mean 1.6. A day with no faults is called a **clear** day.

a. Show that the probability that a day is clear is 0.2019, to 4 decimal places.

After new machinery is installed, the engineer suspects that the mean number of faults per day has increased. She records the number of days, T, up to and including the first clear day.

b. Write down a suitable distribution for T and state an assumption needed.

c. Explain why an increase in the mean number of faults per day would tend to make T larger, and write down suitable hypotheses in terms of p, the probability that a day is clear.

d. Find the critical region for the test at the 5% level of significance.

e. Find the actual significance level of the test.

f. The first clear day after the machinery is installed is the 17th day. State the conclusion of the test in context.

(10 marks)

a) F = faults in a day, $F \sim Po(1.6)$

$P(F = 0) = e^{-1.6}$

$P(F = 0) = 0.201896...$

$\boxed{P(\text{clear}) = 0.2019}$

b) Days until the first clear day

$\boxed{T \sim Geo(p), \text{ assuming days are independent with the same } p}$

c) $p = e^{-\lambda}$ gets smaller as λ gets bigger

Clear days rarer, so the first one tends to come later

$H_0: p = 0.2019 \ (\lambda = 1.6)$

$H_1: p < 0.2019 \ (\lambda > 1.6)$

d) Under H_0, $T \sim Geo(0.2019)$; large T supports H_1

$0.7981^{c-1} \leqslant 0.05$

$c - 1 \geqslant \dfrac{\ln 0.05}{\ln 0.7981} = 13.28...$

$c = 15$

$P(T \geqslant 14) = 0.7981^{13} = 0.0533$

$P(T \geqslant 15) = 0.7981^{14} = 0.0425$

$\boxed{\text{Critical region } T \geqslant 15}$

e) Actual significance level

$\boxed{0.0425}$

f) Is 17 in the critical region?

$17 \geqslant 15$, so reject H_0

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.

Evidence that the mean number of faults per day has increased

Free for classroom and home use: print, photocopy and share. Not for resale or re-hosting.