PGFs of sums and linear functions

Worked solutions

Further Maths topic practice

This topic online

1 The discrete random variable X has probability generating function

$G_X(t) = \frac{1}{5}(1 + 2t + 2t^3)$

The random variable $Y = 2X + 3$

a. Find the probability generating function of Y.

b. Write down $P(Y = 9)$.

c. Find $P(Y > 4)$.

(4 marks)

a) For $Y = aX + b$, $G(t)$ of $Y = t^b \times G(t^a)$ of X

$$G(t) = t^3 \times \frac{1}{5}(1 + 2t^2 + 2(t^2)^3)$$

$$\boxed{G(t) = \frac{1}{5}(t^3 + 2t^5 + 2t^7)}$$

b) Coefficient of t^9

$$\boxed{P(Y = 9) = \frac{2}{5}}$$

c) Powers bigger than 4: t^5 and t^7

$$P(Y > 4) = \frac{2}{5} + \frac{2}{5}$$

$$\boxed{P(Y > 4) = \frac{4}{5}}$$

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2 The independent random variables X and Y are such that $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(5)$.

a. Use probability generating functions to show that $X + Y \sim \text{Po}(7)$.

b. Find $\text{P}(X + Y = 3)$.

(4 marks)

a) Independent, so multiply the pgfs

$$G(t) = e^{2(t-1)} \times e^{5(t-1)}$$

$$G(t) = e^{7(t-1)}$$

This is the pgf of Po(7), and a pgf fixes the distribution

$$\boxed{X + Y \sim \text{Po}(7)}$$

b) Poisson formula with $\lambda = 7$

$$\text{P}(X + Y = 3) = \frac{e^{-7} \times 7^3}{3!}$$

$$\text{P}(X + Y = 3) = 0.052129\ldots$$

$$\boxed{\text{P}(X + Y = 3) = 0.0521}$$

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3 The independent discrete random variables X and Y have probability generating functions

$G_X(t) = \frac{1}{3}(1 + 2t)$ and $G_Y(t) = \frac{1}{4}(1 + t + 2t^2)$

a. Find the probability generating function of $S = X + Y$, giving your answer as a polynomial in t.

b. Find $P(S = 2)$.

c. Find $P(S \geq 2)$.

(4 marks)

a) Independent, so multiply the pgfs

$$G(t) = \frac{1}{3}(1 + 2t) \times \frac{1}{4}(1 + t + 2t^2)$$

$$G(t) = \frac{1}{12}(1 + t + 2t^2 + 2t + 2t^2 + 4t^3)$$

$$\boxed{G(t) = \frac{1}{12}(1 + 3t + 4t^2 + 4t^3)}$$

b) Coefficient of t^2

$$P(S = 2) = \frac{4}{12}$$

$$\boxed{P(S = 2) = \frac{1}{3}}$$

c) Coefficients of t^2 and t^3

$$P(S \geq 2) = \frac{4}{12} + \frac{4}{12}$$

$$\boxed{P(S \geq 2) = \frac{2}{3}}$$

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4 The random variable X has probability generating function

$G_X(t) = k(2 + 3t)^4$

where k is a constant.

a. Find the value of k.

The random variable $W = 3X + 2$

b. Find the probability generating function of W.

c. Find the exact value of $P(W = 11)$.

d. Use calculus and your answer to (b) to find $E(W)$.

(8 marks)

a) $G(1) = 1$

$k(2 + 3)^4 = 1$

$625k = 1$

$$\boxed{k = \frac{1}{625}}$$

b) $Y = aX + b$ gives $t^b \times G(t^a)$: here $a = 3$, $b = 2$

$$G(t) = t^2 \times \frac{1}{625}(2 + 3t^3)^4$$

$$\boxed{G(t) = \frac{t^2(2 + 3t^3)^4}{625}}$$

c) $W = 11$ means $3X + 2 = 11$, so $X = 3$

$$P(X = 3) = \frac{1}{625} \times {}^4C_3 \times 2 \times 3^3$$

$$P(X = 3) = \frac{1}{625} \times 4 \times 2 \times 27$$

$$\boxed{P(W = 11) = \frac{216}{625}}$$

d) Product rule on $G(t) = \frac{1}{625} t^2(2 + 3t^3)^4$

$$G'(t) = \frac{1}{625}\left[2t(2 + 3t^3)^4 + t^2 \times 4(2 + 3t^3)^3 \times 9t^2\right]$$

$$G'(t) = \frac{1}{625}\left[2t(2 + 3t^3)^4 + 36t^4(2 + 3t^3)^3\right]$$

$$E(W) = G'(1) = \frac{1}{625}(2 \times 625 + 36 \times 125)$$

$$E(W) = \frac{1}{625}(1250 + 4500)$$

$$\boxed{E(W) = 9.2}$$

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5 The discrete random variable X has probability generating function $G_X(t)$. The random variable $Y = aX + b$, where a and b are positive integers.

a. Show that $G_Y(t) = t^b G_X(t^a)$

b. Hence, by differentiating, show that $E(Y) = aE(X) + b$

The random variable X has probability generating function $G_X(t) = \frac{1}{16}(3 + t)^2$ and $Y = 4X + 1$

c. Write down $G_Y(t)$ and find $E(Y)$.

(6 marks)

a) Definition of a pgf

$G(t)$ of $Y = E(t^Y)$

$E(t^Y) = E(t^{aX+b})$

$E(t^Y) = E(t^b \times (t^a)^X)$

t^b is a constant

$E(t^Y) = t^b E((t^a)^X)$

$$\boxed{G(t) \text{ of } Y = t^b \times G(t^a) \text{ of } X}$$

b) Product rule and chain rule

$G'(t)$ of $Y = bt^{b-1} G(t^a) + t^b \times at^{a-1} G'(t^a)$

Put $t = 1$: $G(1) = 1$ and $G'(1) = E(X)$

$E(Y) = b \times 1 + a \times E(X)$

$$\boxed{E(Y) = aE(X) + b}$$

c) $a = 4$, $b = 1$

$G(t)$ of $Y = t \times \frac{1}{16}(3 + t^4)^2$

$$\boxed{G(t) \text{ of } Y = \frac{t(3 + t^4)^2}{16}}$$

$E(X)$ from $G(t)$ of X

$G'(t) = \frac{2(3 + t)}{16}$

$E(X) = G'(1) = \frac{8}{16} = \frac{1}{2}$

$E(Y) = 4 \times \frac{1}{2} + 1$

$$\boxed{E(Y) = 3}$$

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6 Ali, Beth and Chen each roll their own fair six-sided dice repeatedly until they get a 5 or a 6. The random variables X_1, X_2 and X_3 are the numbers of rolls they make, and $S = X_1 + X_2 + X_3$

a. Write down the probability generating function of X_1.

b. Find the probability generating function of S, in its simplest form.

c. Hence name the distribution of S, including its parameters.

d. Find the exact value of $P(S = 5)$.

e. Find $E(S)$ and $Var(S)$.

(9 marks)

a) $P(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$, so $X_1 \sim Geo(\frac{1}{3})$

$$G(t) = \frac{\frac{1}{3}t}{(1 - \frac{2}{3}t)}$$

$$\boxed{G(t) = \frac{t}{(3 - 2t)}}$$

b) Independent, so multiply the three pgfs

$$G(t) \text{ of } S = \left(\frac{t}{(3 - 2t)}\right)^3$$

$$\boxed{G(t) \text{ of } S = \frac{t^3}{(3 - 2t)^3}}$$

c) Same form as $\left(\frac{pt}{(1 - qt)}\right)^r$

$$\boxed{S \sim \text{Negative binomial, } r = 3, p = \frac{1}{3}}$$

d) Third success on roll 5: 2 successes in the first 4 rolls

$$P(S = 5) = {}^4C_2 \left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$$

$$P(S = 5) = 6 \times \frac{1}{27} \times \frac{4}{9} = \frac{24}{243}$$

$$\boxed{P(S = 5) = \frac{8}{81}}$$

e) Negative binomial: mean $\frac{r}{p}$, variance $\frac{r(1 - p)}{p^2}$

$$E(S) = \frac{3}{(1/3)} = 9$$

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$$\text{Var}(S) = \frac{3 \times \frac{2}{3}}{\frac{1}{9}} = 18$$

$$\boxed{E(S) = 9,\ \text{Var}(S) = 18}$$

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7 The discrete random variable V has the probability distribution shown.

v	1	2	3
$P(V = v)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

a. Show that the probability generating function of V is $G_V(t) = \frac{t(1 + t)^2}{4}$

The discrete random variable W is independent of V and has probability generating function $G_W(t) = \frac{(1 + t)^3}{8}$

b. Find the probability generating function of $U = V + W$ in its simplest form.

c. Use calculus to find Var(U).

d. Find $P(U = 4)$.

e. State the distribution of $U - 1$, giving its parameters.

(11 marks)

a) $G(t) = \sum P(V = v)\, t^v$

$$G(t) = \frac{1}{4}t + \frac{1}{2}t^2 + \frac{1}{4}t^3$$

$$G(t) = \frac{t}{4}(1 + 2t + t^2)$$

$$\boxed{G(t) \text{ of } V = \frac{t(1 + t)^2}{4}}$$

b) Independent, so multiply

$$G(t) \text{ of } U = \frac{t(1 + t)^2}{4} \times \frac{(1 + t)^3}{8}$$

$$\boxed{G(t) \text{ of } U = \frac{t(1 + t)^5}{32}}$$

c) Product rule

$$G'(t) = \frac{1}{32}\left[(1 + t)^5 + 5t(1 + t)^4\right]$$

$$E(U) = G'(1) = \frac{1}{32}(32 + 80) = \frac{7}{2}$$

$$G''(t) = \frac{1}{32}\left[5(1 + t)^4 + 5(1 + t)^4 + 20t(1 + t)^3\right]$$

$$G''(1) = \frac{1}{32}(80 + 80 + 160) = 10$$

$\text{Var}(U) = G''(1) + G'(1) - [G'(1)]^2$

$$\text{Var}(U) = 10 + \frac{7}{2} - \frac{49}{4}$$

$$\boxed{\text{Var}(U) = \frac{5}{4}}$$

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d) t^4 term: $t \times$ (t^3 term of $(1+t)^5$)

$$P(U=4) = \frac{{}^5C_3}{32} = \frac{10}{32}$$

$$\boxed{P(U=4) = \frac{5}{16}}$$

e) pgf of $U-1$ is $t^{-1} \times G(t)$ of U

$$t^{-1} \times \frac{t(1+t)^5}{32} = \left(\frac{1}{2} + \frac{1}{2}t\right)^5$$

$$\boxed{U-1 \sim B\left(5, \frac{1}{2}\right)}$$

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8 The random variable X has probability generating function

$$G_X(t) = \frac{1}{5 - 4t}$$

a. Use calculus to find Var(X).

The independent random variables X_1 and X_2 each have the same distribution as X, and $Y = X_1 + X_2 + 2$

b. By finding the probability generating function of Y, name the distribution of Y and give its parameters.

c. Hence find $P(X_1 + X_2 > 3)$.

(11 marks)

a) Write as a power

$G(t) = (5 - 4t)^{-1}$

$G'(t) = 4(5 - 4t)^{-2}$

$E(X) = G'(1) = 4$

$G''(t) = 32(5 - 4t)^{-3}$

$G''(1) = 32$

$Var(X) = G''(1) + G'(1) - [G'(1)]^2$

$Var(X) = 32 + 4 - 16$

$\boxed{Var(X) = 20}$

b) Independent copies multiply, and + 2 gives $\times t^2$

$G(t) \text{ of } Y = t^2 \times \frac{1}{(5 - 4t)} \times \frac{1}{(5 - 4t)}$

$G(t) \text{ of } Y = \frac{t^2}{(5 - 4t)^2}$

$G(t) \text{ of } Y = \left(\frac{\frac{1}{5}t}{(1 - \frac{4}{5}t)} \right)^2$

$\boxed{Y \sim \text{Negative binomial, } r = 2, p = \frac{1}{5}}$

c) $X_1 + X_2 > 3$ means $Y > 5$

$P(Y = y) = (y - 1)(0.2)^2(0.8)^{y-2}$

$P(Y \leq 5) = 0.04(1 + 2 \times 0.8 + 3 \times 0.8^2 + 4 \times 0.8^3)$

$P(Y \leq 5) = 0.04 \times 6.568 = 0.26272$

$P(Y > 5) = 1 - 0.26272$

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$P(X_1 + X_2 > 3) = 0.7373$

9 The discrete random variable Y has probability generating function

$G_Y(t) = t^3 e^{4(t^2 - 1)}$

a. Show that $Y = 2X + 3$, where X has a Poisson distribution with a mean that you should state.

b. Use calculus to find E(Y) and Var(Y).

c. Explain why P($Y = 8$) = 0

d. Find the exact value of P($Y = 7$).

(9 marks)

a) Poisson pgf: $e^{\lambda(t-1)}$

$e^{4(t^2-1)} = G(t^2)$ for $X \sim Po(4)$

$G(t)$ of $Y = t^3 \times G(t^2)$ of X

This is the pgf of $2X + 3$

$Y = 2X + 3$ with $X \sim Po(4)$, mean 4

b) Product and chain rules

$G'(t) = 3t^2 e^{4(t^2-1)} + t^3 \times 8t\, e^{4(t^2-1)}$

$G'(t) = (3t^2 + 8t^4)e^{4(t^2-1)}$

$E(Y) = G'(1) = 3 + 8 = 11$

$G''(t) = (6t + 32t^3)e^{4(t^2-1)} + (3t^2 + 8t^4) \times 8te^{4(t^2-1)}$

$G''(1) = 38 + 11 \times 8 = 126$

$Var(Y) = G''(1) + G'(1) - [G'(1)]^2$

$Var(Y) = 126 + 11 - 121$

$E(Y) = 11$, $Var(Y) = 16$

c) Only odd powers of t appear in $G(t)$ of Y

$2X + 3$ is always odd, so $Y = 8$ is impossible

d) $2X + 3 = 7$ gives $X = 2$

$P(Y = 7) = P(X = 2) = \frac{e^{-4} \times 4^2}{2!}$

$P(Y = 7) = 8e^{-4}$

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10 A spinner at a charity fun day scores 1, 2 or 3 with probabilities $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$ respectively. Priya spins it twice, and the scores are independent. T is her total score.

a. Show that the probability generating function of T is $G_T(t) = \frac{t^2(3 + 2t + t^2)^2}{36}$

b. Use calculus to find E(T) and Var(T).

Priya is awarded $S = 5T - 4$ points.

c. Find the probability generating function of S, as a polynomial.

d. Find P($S = 16$) and P($S > 20$).

e. Write down E(S) and Var(S).

(10 marks)

a) pgf of one spin

$$G(t) = \frac{1}{2}t + \frac{1}{3}t^2 + \frac{1}{6}t^3 = \frac{t(3 + 2t + t^2)}{6}$$

Two independent spins, so square it

$$G(t) \text{ of } T = \left(\frac{t(3 + 2t + t^2)}{6}\right)^2$$

$$\boxed{G(t) \text{ of } T = \frac{t^2(3 + 2t + t^2)^2}{36}}$$

b) Expand first

$$(3 + 2t + t^2)^2 = 9 + 12t + 10t^2 + 4t^3 + t^4$$

$$G(t) = \frac{1}{36}(9t^2 + 12t^3 + 10t^4 + 4t^5 + t^6)$$

$$G'(t) = \frac{1}{36}(18t + 36t^2 + 40t^3 + 20t^4 + 6t^5)$$

$$E(T) = G'(1) = \frac{120}{36} = \frac{10}{3}$$

$$G''(t) = \frac{1}{36}(18 + 72t + 120t^2 + 80t^3 + 30t^4)$$

$$G''(1) = \frac{320}{36} = \frac{80}{9}$$

$$Var(T) = \frac{80}{9} + \frac{30}{9} - \frac{100}{9}$$

$$\boxed{E(T) = \frac{10}{3},\ Var(T) = \frac{10}{9}}$$

c) $S = 5T - 4$: $t^{-4} \times G(t^5)$

$$G(t) \text{ of } S = t^{-4} \times \frac{1}{36}(9t^{10} + 12t^{15} + 10t^{20} + 4t^{25} + t^{30})$$

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$$G(t) \text{ of } S = \frac{1}{36}(9t^6 + 12t^{11} + 10t^{16} + 4t^{21} + t^{26})$$

d) Read the coefficients

$$P(S = 16) = \frac{10}{36}$$

$$\boxed{P(S = 16) = \frac{5}{18}}$$

$$P(S > 20) = \frac{4}{36} + \frac{1}{36}$$

$$\boxed{P(S > 20) = \frac{5}{36}}$$

e) $E(aT + b) = aE(T) + b$ and $Var(aT + b) = a^2 Var(T)$

$$E(S) = 5 \times \frac{10}{3} - 4 = \frac{38}{3}$$

$$Var(S) = 25 \times \frac{10}{9} = \frac{250}{9}$$

$$\boxed{E(S) = \frac{38}{3}, Var(S) = \frac{250}{9}}$$

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11 An office manager, Tomasz, receives urgent emails at a mean rate of 1.5 per hour and routine emails at a mean rate of 2.5 per hour, independently and at random. In a randomly chosen hour, X is the number of urgent emails and Y is the number of routine emails.

a. Use probability generating functions to find the distribution of $X + Y$, and hence find the exact probability that Tomasz receives 3 emails in a randomly chosen hour.

Urgent emails count 2 points each and routine emails 1 point each towards a workload score $Z = 2X + Y$.

b. Show that the probability generating function of Z is $G_Z(t) = e^{1.5t^2 + 2.5t - 4}$

c. Use calculus to find E(Z) and Var(Z).

d. Explain why Z does not have a Poisson distribution.

e. Find P($Z = 2$), giving your answer to 4 decimal places.

(11 marks)

a) Independent, so multiply the Poisson pgfs

$$G(t) = e^{1.5(t-1)} \times e^{2.5(t-1)} = e^{4(t-1)}$$

This is the pgf of Po(4)

$$P(X + Y = 3) = \frac{e^{-4} \times 4^3}{3!}$$

$$P(X + Y = 3) = \frac{32}{3}e^{-4}$$

$$X + Y \sim Po(4), P = \frac{32}{3}e^{-4} = 0.1954$$

b) pgf of $2X$ is $G(t^2)$ of X

$$G(t) \text{ of } Z = e^{1.5(t^2-1)} \times e^{2.5(t-1)}$$

$$G(t) \text{ of } Z = e^{1.5t^2 - 1.5 + 2.5t - 2.5}$$

$$G(t) \text{ of } Z = e^{1.5t^2 + 2.5t - 4}$$

c) Chain rule

$$G'(t) = (3t + 2.5)e^{1.5t^2 + 2.5t - 4}$$

$$E(Z) = G'(1) = 5.5 \times e^0 = 5.5$$

Product rule

$$G''(t) = 3e^{1.5t^2 + 2.5t - 4} + (3t + 2.5)^2 e^{1.5t^2 + 2.5t - 4}$$

$$G''(1) = 3 + 30.25 = 33.25$$

$$Var(Z) = 33.25 + 5.5 - 5.5^2$$

$$E(Z) = 5.5, Var(Z) = 8.5$$

d) Poisson needs mean = variance

$5.5 \neq 8.5$, so Z is not Poisson

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e) $Z = 2$: $(X = 0, Y = 2)$ or $(X = 1, Y = 0)$

$$P(Z = 2) = e^{-1.5} \times e^{-2.5} \times \frac{2.5^2}{2} + 1.5e^{-1.5} \times e^{-2.5}$$

$$P(Z = 2) = e^{-4}(3.125 + 1.5)$$

$$P(Z = 2) = \frac{37}{8}e^{-4} = 0.084709...$$

$$\boxed{P(Z = 2) = 0.0847}$$

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12 The independent random variables X and Y have probability generating functions

$G_X(t) = (a + bt)^2$ and $G_Y(t) = \frac{t}{3 - 2t}$

where a and b are positive constants. The random variable $W = X + Y$

a. Write down the probability generating function of W.

b. Given that $P(W = 1) = \frac{1}{12}$, find the values of a and b.

c. Use calculus to find $E(W)$ and $Var(W)$.

d. Find the exact value of $P(W = 3)$.

e. Find the probability generating function of $3W + 1$

(12 marks)

a) Independent, so multiply

$$G(t) \text{ of } W = \frac{t(a + bt)^2}{(3 - 2t)}$$

b) Lowest power is t: set t' coefficient

$$G(t) \text{ of } Y = \frac{t}{3} \times (1 - \frac{2}{3}t)^{-1} = \frac{t}{3} + \frac{2}{9}t^2 + \ldots$$

$$P(W = 1) = a^2 \times \frac{1}{3} = \frac{1}{12}$$

$$a^2 = \frac{1}{4}$$

$$a = \frac{1}{2}$$

$G(1)$ of $X = 1$ and $a, b > 0$

$$a + b = 1$$

$$a = \frac{1}{2}, b = \frac{1}{2}$$

c) Find $G'(1)$ and $G''(1)$ of each part

$$G(t) \text{ of } X = \frac{1}{4}(1 + t)^2$$

$$G'(1) \text{ of } X = \frac{1}{2}(1 + 1) = 1$$

$$G''(1) \text{ of } X = \frac{1}{2}$$

$$G(t) \text{ of } Y = t(3 - 2t)^{-1}$$

$$G'(t) \text{ of } Y = (3 - 2t)^{-1} + 2t(3 - 2t)^{-2} = 3(3 - 2t)^{-2}$$

$$G'(1) \text{ of } Y = 3$$

$$G''(t) \text{ of } Y = 12(3 - 2t)^{-3}$$

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$G''(1)$ of $Y = 12$

Product rule on $G(t)$ of W, then put $t = 1$ (both pgfs equal 1 there)

$$E(W) = 1 + 3 = 4$$

$$G''(1) \text{ of } W = \frac{1}{2} + 2 \times 1 \times 3 + 12 = \frac{37}{2}$$

$$\text{Var}(W) = \frac{37}{2} + 4 - 16$$

$$\boxed{E(W) = 4, \text{Var}(W) = \frac{13}{2}}$$

d) $G(t) = \frac{1}{4}(1 + 2t + t^2) \times \frac{t}{3}(1 + \frac{2}{3}t + \frac{4}{9}t^2 + \ldots)$

t^3 needs t^2 from the last two brackets combined

$$P(W = 3) = \frac{1}{12}(1 \times \frac{4}{9} + 2 \times \frac{2}{3} + 1 \times 1)$$

$$P(W = 3) = \frac{1}{12} \times \frac{25}{9}$$

$$\boxed{P(W = 3) = \frac{25}{108}}$$

e) $3W + 1$: $t \times G(t^3)$ of W

$$G(t) = t \times \frac{t^3(1 + t^3)^2}{4(3 - 2t^3)}$$

$$\boxed{G(t) = \frac{t^4(1 + t^3)^2}{4(3 - 2t^3)}}$$

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