Probability generating functions

Worked solutions

Further Maths topic practice

This topic online

1 The discrete random variable X has the probability distribution shown.

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{4}$

a. Write down the probability generating function of X.

b. Use calculus to find $E(X)$.

(4 marks)

a) $G(t) = E(t^X) = \sum P(X = x)\, t^x$

$$G(t) = \frac{1}{8} + \frac{3}{8}t + \frac{1}{4}t^2 + \frac{1}{4}t^3$$

$$\boxed{G(t) = \frac{1}{8} + \frac{3}{8}t + \frac{1}{4}t^2 + \frac{1}{4}t^3}$$

b) Differentiate, then $E(X) = G'(1)$

$$G'(t) = \frac{3}{8} + \frac{1}{2}t + \frac{3}{4}t^2$$

$$E(X) = G'(1) = \frac{3}{8} + \frac{1}{2} + \frac{3}{4}$$

$$\boxed{E(X) = \frac{13}{8}}$$

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2 The discrete random variable X has probability generating function

$G_X(t) = k(t + 2t^2 + 4t^4)$

where k is a constant.

a. Find the value of k.

b. Write down $P(X = 2)$.

c. Find $P(X \geq 2)$.

d. Write down $P(X = 3)$, giving a reason.

(4 marks)

a) Probabilities add to 1, so $G(1) = 1$

$k(1 + 2 + 4) = 1$

$7k = 1$

$\boxed{k = \frac{1}{7}}$

b) $P(X = 2)$ = coefficient of t^2

$\boxed{P(X = 2) = \frac{2}{7}}$

c) Add the coefficients of t^2 and t^4

$P(X \geq 2) = \frac{2}{7} + \frac{4}{7}$

$\boxed{P(X \geq 2) = \frac{6}{7}}$

d) There is no t^3 term

$\boxed{P(X = 3) = 0}$

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3 Write down the probability generating function of each of these random variables.

a. $A \sim B(8, 0.3)$

b. $B \sim Po(2.5)$

c. $C \sim Geo(\frac{1}{4})$

Each of these is the probability generating function of a standard distribution. Name the distribution, including its parameters.

d. $G_D(t) = e^{3t-3}$

e. $G_V(t) = \frac{(2+t)^5}{243}$

(5 marks)

a) Binomial: $G(t) = (1 - p + pt)^n$

$$G(t) = (0.7 + 0.3t)^8$$

b) Poisson: $G(t) = e^{\lambda(t-1)}$

$$G(t) = e^{2.5(t-1)}$$

c) Geometric: $G(t) = \frac{pt}{(1-(1-p)t)}$

$$G(t) = \frac{\frac{1}{4}t}{(1-\frac{3}{4}t)}$$

$$G(t) = \frac{t}{(4-3t)}$$

d) Take out the 3

$$e^{3t-3} = e^{3(t-1)}$$

$$D \sim Po(3)$$

e) $243 = 3^5$

$$\frac{(2+t)^5}{243} = (\frac{2}{3} + \frac{1}{3}t)^5$$

$$V \sim B(5, \frac{1}{3})$$

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4 The discrete random variable X has probability generating function

$G_X(t) = k(1 + 3t^2)^3$

where k is a constant.

a. Show that $k = \frac{1}{64}$

b. Find $P(X = 4)$.

c. Use calculus to show that $\text{Var}(X) = \frac{9}{4}$

(9 marks)

a) $G(1) = 1$

$k(1 + 3)^3 = 1$

$64k = 1$

$\boxed{k = \frac{1}{64}}$

b) Expand with the binomial theorem

$(1 + 3t^2)^3 = 1 + 3(3t^2) + 3(3t^2)^2 + (3t^2)^3$

$(1 + 3t^2)^3 = 1 + 9t^2 + 27t^4 + 27t^6$

$P(X = 4) =$ coefficient of t^4

$\boxed{P(X = 4) = \frac{27}{64}}$

c) Chain rule

$G'(t) = \frac{1}{64} \times 3(1 + 3t^2)^2 \times 6t$

$G'(t) = \frac{18t(1 + 3t^2)^2}{64}$

$E(X) = G'(1) = \frac{18 \times 16}{64} = \frac{9}{2}$

Product rule

$G''(t) = \frac{1}{64}[18(1 + 3t^2)^2 + 18t \times 2(1 + 3t^2) \times 6t]$

$G''(t) = \frac{1}{64}[18(1 + 3t^2)^2 + 216t^2(1 + 3t^2)]$

$G''(1) = \frac{1}{64}(288 + 864) = 18$

$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$

$\text{Var}(X) = 18 + \frac{9}{2} - (\frac{9}{2})^2$

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$$\text{Var}(X) = 18 + \frac{9}{2} - \frac{81}{4}$$

$$\boxed{\text{Var}(X) = \frac{9}{4}}$$

5 The discrete random variable X takes only the values 1, 2 and 3, and has probability generating function

$G_X(t) = at + bt^2 + ct^3$

where a, b and c are constants. Given that $E(X) = 2.1$ and $\text{Var}(X) = 0.69$,

a. show that $G_X''(1) = 3$

b. find the values of a, b and c.

(7 marks)

a) Rearrange $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$

$$G''(1) = \text{Var}(X) - G'(1) + [G'(1)]^2$$

$$G''(1) = 0.69 - 2.1 + 2.1^2$$

$$G''(1) = 0.69 - 2.1 + 4.41$$

$$\boxed{G''(1) = 3}$$

b) Three equations: $G(1) = 1$, $G'(1) = 2.1$, $G''(1) = 3$

$$a + b + c = 1$$

$$G'(t) = a + 2bt + 3ct^2$$

$$a + 2b + 3c = 2.1$$

$$G''(t) = 2b + 6ct$$

$$2b + 6c = 3$$

Subtract the first equation from the second

$$b + 2c = 1.1$$

Halve the third equation

$$b + 3c = 1.5$$

Subtract

$$c = 0.4$$

$$b = 1.1 - 0.8 = 0.3$$

$$a = 1 - 0.3 - 0.4 = 0.3$$

$$\boxed{a = 0.3,\ b = 0.3,\ c = 0.4}$$

6 The discrete random variable X has probability generating function

$$G_X(t) = k \ln\left(\frac{3}{3 - 2t}\right)$$

where k is a constant.

a. Find the exact value of k.

b. Find the exact value of $E(X)$.

c. Find the exact value of $\text{Var}(X)$.

d. Find $P(X = 3)$, giving your answer in the form $\frac{p}{q \ln 3}$ where p and q are integers.

(10 marks)

a) $G(1) = 1$

$$k \ln\left(\frac{3}{(3 - 2)}\right) = 1$$

$$k \ln 3 = 1$$

$$\boxed{k = \frac{1}{\ln 3}}$$

b) Use a log law first

$$G(t) = k[\ln 3 - \ln(3 - 2t)]$$

$$G'(t) = k \times \frac{2}{(3 - 2t)}$$

$$E(X) = G'(1) = 2k$$

$$\boxed{E(X) = \frac{2}{\ln 3}}$$

c) Differentiate again

$$G'(t) = 2k(3 - 2t)^{-1}$$

$$G''(t) = 4k(3 - 2t)^{-2}$$

$$G''(1) = 4k = \frac{4}{\ln 3}$$

$$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

$$\text{Var}(X) = \frac{4}{\ln 3} + \frac{2}{\ln 3} - \frac{4}{(\ln 3)^2}$$

$$\boxed{\text{Var}(X) = \frac{6}{\ln 3} - \frac{4}{(\ln 3)^2}}$$

d) Series for $\ln(1 + x)$ with $x = -\frac{2t}{3}$

$$\ln\left(\frac{3}{(3 - 2t)}\right) = -\ln\left(1 - \frac{2t}{3}\right)$$

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$$-\ln\left(1-\frac{2t}{3}\right) = \frac{2t}{3} + \frac{1}{2}\left(\frac{2t}{3}\right)^2 + \frac{1}{3}\left(\frac{2t}{3}\right)^3 + \ldots$$

Coefficient of t^3

$$\frac{1}{3} \times \frac{8}{27} = \frac{8}{81}$$

$$P(X=3) = k \times \frac{8}{81}$$

$$\boxed{P(X=3) = \frac{8}{(81 \ln 3)}}$$

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7 The random variable $X \sim \text{Geo}(p)$, so that $P(X = x) = p(1 - p)^{x-1}$ for $x = 1, 2, 3, \ldots$

a. Show that the probability generating function of X is $G_X(t) = \dfrac{pt}{1 - (1 - p)t}$

b. Use calculus to show that $E(X) = \dfrac{1}{p}$ and $\text{Var}(X) = \dfrac{1 - p}{p^2}$

(8 marks)

a) Let $q = 1 - p$

$$G(t) = \sum P(X = x)\, t^x$$

$$G(t) = pt + pqt^2 + pq^2t^3 + \ldots$$

$$G(t) = pt[1 + qt + (qt)^2 + \ldots]$$

Geometric series, first term pt, ratio qt ($|qt| < 1$)

$$G(t) = \frac{pt}{(1 - qt)}$$

$$\boxed{G(t) = \frac{pt}{(1 - (1 - p)t)}}$$

b) Quotient rule

$$G'(t) = \frac{(p(1 - qt) - pt(-q))}{(1 - qt)^2}$$

$$G'(t) = \frac{p}{(1 - qt)^2}$$

$1 - q = p$

$$E(X) = G'(1) = \frac{p}{p^2} = \frac{1}{p}$$

$$G'(t) = p(1 - qt)^{-2}$$

$$G''(t) = 2pq(1 - qt)^{-3}$$

$$G''(1) = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$

$$\text{Var}(X) = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$\text{Var}(X) = \frac{(2q + p - 1)}{p^2}$$

$p - 1 = -q$

$$\text{Var}(X) = \frac{q}{p^2}$$

$$\boxed{\text{Var}(X) = \frac{(1 - p)}{p^2}}$$

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8 The discrete random variable X has probability generating function

$$G_X(t) = \frac{t^3}{(4 - 3t)^3}$$

a. Specify fully the distribution of X.

A spinner has four equal sectors, one of which is red. The spinner is spun repeatedly.

b. Describe an outcome that could be modelled by X.

c. Use calculus to find $E(X)$ and $Var(X)$.

d. Find the exact value of $P(X = 5)$.

(11 marks)

a) Divide top and bottom by 4^3

$$G(t) = \left(\frac{t}{(4 - 3t)}\right)^3 = \left(\frac{\frac{1}{4}t}{(1 - \frac{3}{4}t)}\right)^3$$

Negative binomial pgf: $\left(\frac{pt}{(1 - qt)}\right)^r$

$$X \sim \text{Negative binomial, } r = 3, p = \frac{1}{4}$$

b) $P(\text{red}) = \frac{1}{4}$ on each spin

X = the number of spins needed to get the third red

c) Product rule with $G(t) = t^3(4 - 3t)^{-3}$

$$G'(t) = 3t^2(4 - 3t)^{-3} + 9t^3(4 - 3t)^{-4}$$

$$E(X) = G'(1) = 3 + 9 = 12$$

$$G''(t) = 6t(4 - 3t)^{-3} + 27t^2(4 - 3t)^{-4} + 27t^2(4 - 3t)^{-4} + 108t^3(4 - 3t)^{-5}$$

$$G''(1) = 6 + 27 + 27 + 108 = 168$$

$Var(X) = G''(1) + G'(1) - [G'(1)]^2$

$$Var(X) = 168 + 12 - 144$$

$$E(X) = 12, Var(X) = 36$$

d) 3rd red on spin 5: 2 reds in the first 4 spins, then red

$$P(X = 5) = {}^4C_2 \left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$$

$$P(X = 5) = 6 \times \frac{1}{64} \times \frac{9}{16}$$

$$P(X = 5) = \frac{27}{512}$$

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9 The discrete random variable X has probability function

$$P(X = r) = \frac{r + 1}{2^{r+2}} \quad r = 0, 1, 2, \ldots$$

a. Show that the probability generating function of X is $G_X(t) = \frac{1}{(2 - t)^2}$

b. Use calculus to find $E(X)$ and $\text{Var}(X)$. (8 marks)

a) $G(t) = \sum P(X = r)\, t^r$

$$G(t) = \frac{1}{4} + \frac{2}{8}t + \frac{3}{16}t^2 + \frac{4}{32}t^3 + \ldots$$

$$G(t) = \frac{1}{4}\left[1 + 2\left(\frac{t}{2}\right) + 3\left(\frac{t}{2}\right)^2 + 4\left(\frac{t}{2}\right)^3 + \ldots\right]$$

Binomial expansion: $(1 - x)^{-2} = 1 + 2x + 3x^2 + \ldots$

$$G(t) = \frac{1}{4}\left(1 - \frac{t}{2}\right)^{-2}$$

$$G(t) = \frac{1}{4} \times \left(2 - t\right)^{2}$$

$$\boxed{G(t) = \frac{1}{(2 - t)^2}}$$

b) $G(t) = (2 - t)^{-2}$

$$G'(t) = 2(2 - t)^{-3}$$

$$E(X) = G'(1) = 2$$

$$G''(t) = 6(2 - t)^{-4}$$

$$G''(1) = 6$$

$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$

$$\text{Var}(X) = 6 + 2 - 4$$

$$\boxed{E(X) = 2,\ \text{Var}(X) = 4}$$

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10 The random variable $X \sim B(n, p)$ has probability generating function $G_X(t) = (q + pt)^n$, where $q = 1 - p$.

a. Use calculus to show that $E(X) = np$ and $Var(X) = npq$.

A bakery puts n biscuits in each tin. Each biscuit, independently, has probability p of being broken. The number of broken biscuits in a tin has mean 4.8 and variance 1.92.

b. Find the values of n and p.

c. Find the probability that a randomly chosen tin contains exactly 6 broken biscuits. (10 marks)

a) Chain rule

$G'(t) = np(q + pt)^{n-1}$

$q + p = 1$

$E(X) = G'(1) = np$

$G''(t) = n(n-1)p^2(q + pt)^{n-2}$

$G''(1) = n(n-1)p^2$

$Var(X) = G''(1) + G'(1) - [G'(1)]^2$

$Var(X) = n^2p^2 - np^2 + np - n^2p^2$

$Var(X) = np - np^2 = np(1 - p)$

$\boxed{Var(X) = npq}$

b) Divide the variance by the mean

$\frac{npq}{np} = \frac{1.92}{4.8}$

$q = 0.4$

$p = 0.6$

$n = \frac{4.8}{0.6}$

$\boxed{n = 8, p = 0.6}$

c) $X \sim B(8, 0.6)$

$P(X = 6) = {}^8C_6 (0.6)^6 (0.4)^2$

$P(X = 6) = 28 \times 0.046656 \times 0.16$

$P(X = 6) = 0.209018...$

$\boxed{P(X = 6) = 0.2090}$

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11 In a game at a school fair, Bilal rolls a fair six-sided dice until he gets a six, but he is allowed at most 3 rolls. The random variable R is the number of rolls he makes.

a. Show that the probability generating function of R is $G_R(t) = \frac{1}{36}(6t + 5t^2 + 25t^3)$

b. Use calculus to find the exact values of $E(R)$ and $Var(R)$.

Each roll costs 20p.

c. Find the expected amount Bilal pays to play the game. (9 marks)

a) Stop on a six, or after the 3rd roll

$$P(R = 1) = \frac{1}{6} = \frac{6}{36}$$

$$P(R = 2) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

Third roll happens if the first two are not sixes

$$P(R = 3) = \left(\frac{5}{6}\right)^2 = \frac{25}{36}$$

$$\boxed{G(t) = \frac{1}{36}(6t + 5t^2 + 25t^3)}$$

b) Differentiate

$$G'(t) = \frac{1}{36}(6 + 10t + 75t^2)$$

$$E(R) = G'(1) = \frac{91}{36}$$

$$G''(t) = \frac{1}{36}(10 + 150t)$$

$$G''(1) = \frac{160}{36} = \frac{40}{9}$$

$$Var(R) = G''(1) + G'(1) - [G'(1)]^2$$

$$Var(R) = \frac{40}{9} + \frac{91}{36} - \left(\frac{91}{36}\right)^2$$

$$Var(R) = \frac{5760}{1296} + \frac{3276}{1296} - \frac{8281}{1296}$$

$$\boxed{E(R) = \frac{91}{36},\ Var(R) = \frac{755}{1296}}$$

c) Cost = 20R pence

$$E(20R) = 20 \times \frac{91}{36}$$

$$E(20R) = 50.555\ldots$$

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50.6p

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12 The discrete random variable X has probability generating function

$G_X(t) = (a + bt + ct^2)^2$

where a, b and c are positive constants.

a. Explain why $a + b + c = 1$

Given that $P(X = 0) = \frac{1}{16}$ and $E(X) = 2.5$,

b. find the values of a, b and c

c. find $P(X = 2)$

d. use calculus to find $Var(X)$.

(12 marks)

a) Probabilities add to 1, so $G(1) = 1$

$(a + b + c)^2 = 1$

$\boxed{a + b + c = 1 \text{ (it is positive, so not } -1)}$

b) $P(X = 0) = G(0)$

$a^2 = \frac{1}{16}$

$a = \frac{1}{4}$

Chain rule

$G'(t) = 2(a + bt + ct^2)(b + 2ct)$

$G'(1) = 2(1)(b + 2c) = 2.5$

$b + 2c = 1.25$

$b + c = 1 - \frac{1}{4} = 0.75$

Subtract

$c = 0.5$

$b = 0.25$

$\boxed{a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{1}{2}}$

c) $G(t) = \frac{1}{16}(1 + t + 2t^2)^2$

$(1 + t + 2t^2)^2 = 1 + 2t + t^2 + 4t^2 + 4t^3 + 4t^4$

$(1 + t + 2t^2)^2 = 1 + 2t + 5t^2 + 4t^3 + 4t^4$

$\boxed{P(X = 2) = \frac{5}{16}}$

d) Product rule on $G'(t) = \frac{1}{16} \times 2(1 + t + 2t^2)(1 + 4t)$

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$$G''(t) = \frac{1}{16}\left[2(1+4t)^2 + 8(1+t+2t^2)\right]$$

$$G''(1) = \frac{1}{16}(50+32) = \frac{41}{8}$$

$$\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

$$\text{Var}(X) = \frac{41}{8} + 2.5 - 6.25$$

$$\boxed{\text{Var}(X) = \frac{11}{8}}$$

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