The geometric distribution

Worked solutions

Further Maths topic practice

This topic online

1 The random variable $X \sim \text{Geo}(0.3)$.

Find

a. $P(X = 4)$

b. $P(X > 5)$

c. $P(X \leq 3)$

(4 marks)

a) $P(X = x) = (1 - p)^{x-1} p$

$P(X = 4) = 0.7^3 \times 0.3$

$P(X = 4) = 0.343 \times 0.3$

$\boxed{P(X = 4) = 0.1029}$

b) $X > 5$ means the first 5 trials all fail

$P(X > 5) = 0.7^5$

$\boxed{P(X > 5) = 0.16807 \ (0.1681)}$

c) Use $P(X \leq x) = 1 - (1 - p)^x$

$P(X \leq 3) = 1 - 0.7^3$

$P(X \leq 3) = 1 - 0.343$

$\boxed{P(X \leq 3) = 0.657}$

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2 Bethan rolls a fair eight-sided dice, numbered 1 to 8, until she rolls an 8. The random variable X is the number of rolls she makes.

a. Write down a suitable distribution for X, stating the value of its parameter, and give two conditions needed for this distribution to be a suitable model.

b. Find $P(X = 5)$.

c. Find $E(X)$ and $Var(X)$.

(6 marks)

a) Number of trials up to and including the first success

$$X \sim Geo\left(\frac{1}{8}\right)$$

The rolls are independent

P(rolling an 8) stays $\frac{1}{8}$ on every roll

b) Four non-8s then an 8

$$P(X = 5) = \left(\frac{7}{8}\right)^4 \times \frac{1}{8}$$

$$P(X = 5) = \frac{2401}{32768}$$

$$P(X = 5) = 0.0733$$

c) $E(X) = \frac{1}{p}$, $Var(X) = \frac{1-p}{p^2}$

$$E(X) = 1 \div \frac{1}{8} = 8$$

$$Var(X) = \frac{7}{8} \div \frac{1}{64}$$

$$Var(X) = \frac{7}{8} \times 64$$

$$E(X) = 8,\ Var(X) = 56$$

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3 The random variable Y has a geometric distribution with $E(Y) = 2.5$

a. Find the parameter of the distribution.

b. Find $P(3 \leq Y \leq 7)$.

c. Find the standard deviation of Y.

(5 marks)

a) $E(Y) = \frac{1}{p}$

$\frac{1}{p} = 2.5$

$p = 0.4$

b) $P(Y > y) = (1 - p)^y$

$P(3 \leq Y \leq 7) = P(Y > 2) - P(Y > 7)$

$P(3 \leq Y \leq 7) = 0.6^2 - 0.6^7$

$P(3 \leq Y \leq 7) = 0.36 - 0.0279936$

$P(3 \leq Y \leq 7) = 0.3320$

c) $Var(Y) = \frac{1 - p}{p^2}$

$Var(Y) = \frac{0.6}{0.16} = 3.75$

$sd = \sqrt{3.75}$

$sd = 1.94$ (3 s.f.)

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4 The random variable $R \sim \text{Geo}(p)$, where $p < 0.5$

Given that $P(R = 2) = 0.1875$

a. show that $p = 0.25$

b. find $P(R \geq 5)$

c. find $E(R^2)$.

(7 marks)

a) One failure then a success

$$(1 - p)p = 0.1875$$

$$p^2 - p + 0.1875 = 0$$

$$(p - 0.25)(p - 0.75) = 0$$

$p = 0.25$ or $p = 0.75$

$\boxed{p < 0.5 \text{ so } p = 0.25}$

b) $R \geq 5$ means the first 4 trials fail

$$P(R \geq 5) = 0.75^4$$

$\boxed{P(R \geq 5) = 0.3164}$

c) Rearrange $\text{Var}(R) = E(R^2) - [E(R)]^2$

$$E(R) = \frac{1}{0.25} = 4$$

$$\text{Var}(R) = \frac{0.75}{0.25^2} = 12$$

$$E(R^2) = \text{Var}(R) + [E(R)]^2$$

$$E(R^2) = 12 + 16$$

$\boxed{E(R^2) = 28}$

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5 Owen works at a call centre selling home insurance. The probability that any one of his calls results in a sale is 0.12, independently of all other calls.

a. Find the probability that his first sale of the day comes on his 6th call.

b. Find the smallest number of calls, n, that Owen needs to make so that the probability of at least one sale in his first n calls is greater than 0.95

(6 marks)

$X \sim Geo(0.12)$

a) Five calls without a sale, then a sale

$P(X = 6) = 0.88^5 \times 0.12$

$P(X = 6) = 0.527731... \times 0.12$

$\boxed{P(X = 6) = 0.0633}$

b) At least one sale in n calls means $X \leq n$

$P(X \leq n) = 1 - 0.88^n$

$1 - 0.88^n > 0.95$

$0.88^n < 0.05$

$n \ln 0.88 < \ln 0.05$

ln 0.88 is negative, so the inequality flips

$$n > \frac{\ln 0.05}{\ln 0.88}$$

$n > 23.43...$

Check: $1 - 0.88^{23} = 0.9471$, $1 - 0.88^{24} = 0.9535$

$\boxed{n = 24}$

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6 Kwame and Lily take turns to throw a dart at a small target. Kwame throws first. The first person to hit the target wins the game. Each time Kwame throws he hits the target with probability 0.2, and each time Lily throws she hits the target with probability 0.3, independently of all other throws.

a. Find the probability that the game ends on the 5th throw.

b. Find the probability that Lily wins the game on her 3rd throw.

c. Show that the probability that Kwame wins the game is $\frac{5}{11}$

(7 marks)

a) Throw 5 is Kwame's 3rd throw: 4 misses then a hit

$$P = 0.8 \times 0.7 \times 0.8 \times 0.7 \times 0.2$$

$$P = 0.06272$$

$$\boxed{0.0627}$$

b) Lily's 3rd throw is throw 6

Both miss in a round: $0.8 \times 0.7 = 0.56$

$$P = 0.56^2 \times 0.8 \times 0.3$$

$$P = 0.075264$$

$$\boxed{0.0753}$$

c) Kwame wins on his 1st, 2nd, 3rd, ... throw

$$P(K) = 0.2 + 0.56 \times 0.2 + 0.56^2 \times 0.2 + \ldots$$

Infinite geometric series, $a = 0.2$, $r = 0.56$

$$P(K) = \frac{0.2}{1 - 0.56}$$

$$P(K) = \frac{0.2}{0.44}$$

$$\boxed{P(\text{Kwame wins}) = \frac{5}{11}}$$

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7 A quality inspector tests light bulbs from a production line one at a time. Each bulb is faulty with probability 0.08, independently of other bulbs. The random variable X is the number of bulbs tested up to and including the first faulty bulb.

a. Find $P(X > 12 \mid X > 5)$.

b. Find $P(X = 9 \mid X > 5)$.

c. Explain, in the context of the question, why your answer to part (a) is equal to $P(X > 7)$. (6 marks)

$X \sim Geo(0.08)$, $P(X > x) = 0.92^x$

a) $X > 12$ is inside $X > 5$

$$P(X > 12 \mid X > 5) = \frac{P(X > 12)}{P(X > 5)}$$

$$P(X > 12 \mid X > 5) = \frac{0.92^{12}}{0.92^5}$$

$$P(X > 12 \mid X > 5) = 0.92^7$$

$\boxed{0.5578}$

b) $X = 9$ is inside $X > 5$

$$P(X = 9 \mid X > 5) = \frac{0.92^8 \times 0.08}{0.92^5}$$

$$P(X = 9 \mid X > 5) = 0.92^3 \times 0.08$$

$$P(X = 9 \mid X > 5) = 0.06229...$$

$\boxed{0.0623}$

c) Lack of memory

The first 5 bulbs were fine, but later bulbs are independent and each is still faulty with probability 0.08 so counting starts again from bulb 6

$\boxed{X > 12 \text{ means more than 7 more bulbs, same as } P(X > 7)}$

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8 A spinner has four equal sectors, one of which is red. Anika spins the spinner until it lands on red. The random variable A is the number of spins she makes. Ben throws a fair coin until it shows heads. The random variable C is the number of throws he makes. A and C are independent.

a. Find P(A = 2).

b. Find P($\{A = 2\} \cup \{C = 2\}$).

c. Show that P($A = C$) = $\frac{1}{5}$

(8 marks)

$A \sim Geo(\frac{1}{4}), C \sim Geo(\frac{1}{2})$

a) Miss then red

$$P(A = 2) = \frac{3}{4} \times \frac{1}{4}$$

$$\boxed{P(A = 2) = \frac{3}{16}}$$

b) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

$$P(C = 2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Independent, so multiply for the intersection

$$P(A = 2 \text{ and } C = 2) = \frac{3}{16} \times \frac{1}{4} = \frac{3}{64}$$

$$P(\text{union}) = \frac{3}{16} + \frac{1}{4} - \frac{3}{64}$$

$$P(\text{union}) = \frac{12}{64} + \frac{16}{64} - \frac{3}{64}$$

$$\boxed{P(\{A = 2\} \cup \{C = 2\}) = \frac{25}{64}}$$

c) Add $P(A = k \text{ and } C = k)$ for $k = 1, 2, 3, \ldots$

$$P(A = C) = \sum \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right) \times \left(\frac{1}{2}\right)^{k}$$

$$P(A = C) = \sum \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{k-1}$$

$$P(A = C) = \frac{1}{8} \sum \left(\frac{3}{8}\right)^{k-1}$$

Geometric series, $a = 1$, $r = \frac{3}{8}$, sum to infinity

$$P(A = C) = \frac{1}{8} \times \frac{1}{1 - 3/8}$$

$$P(A = C) = \frac{1}{8} \times \frac{8}{5}$$

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$$P(A = C) = \frac{1}{5}$$

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9 In a game, a fair six-sided dice is rolled until it shows a 5 or a 6. The random variable X is the number of rolls needed.

a. Find E(X) and Var(X).

Before playing a long series of these games, Ruth must choose her score for every game. Her score will either be X^2 for every game, or $5X - 1$ for every game. She wants the higher total score over the series.

b. Showing your calculations, state which score Ruth should choose.

c. Find the probability that, in a single game, X^2 is greater than $5X - 1$

(8 marks)

a) $P(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$

$X \sim Geo(\frac{1}{3})$

$E(X) = \frac{1}{p} = 3$

$Var(X) = \frac{2}{3} \div \frac{1}{9}$

$\boxed{E(X) = 3,\ Var(X) = 6}$

b) Compare the expected score per game

$E(X^2) = Var(X) + [E(X)]^2$

$E(X^2) = 6 + 9 = 15$

$E(5X - 1) = 5 \times 3 - 1 = 14$

$15 > 14$, and over many games the mean score is close to E

$\boxed{\text{Choose } X^2}$

c) Solve the inequality

$X^2 > 5X - 1$

$X^2 - 5X + 1 > 0$

Roots of $X^2 - 5X + 1 = 0$

$X = \frac{5 \pm \sqrt{21}}{2} = 0.209 \text{ or } 4.79$

So $X < 0.209$ or $X > 4.79$

X is a whole number, at least 1

$X \geq 5$

$P(X \geq 5) = (\frac{2}{3})^4$

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$\frac{16}{81} = 0.1975$

10 In an archery practice session, Hana shoots arrows until she first hits the gold ring. The probability that any arrow hits the gold is p, independently of all other arrows. The random variable X is the number of arrows she shoots in a session. The probability that she hits the gold within her first 2 arrows is 0.64

a. Show that $p = 0.4$

b. Find $P(X > 3)$.

Hana has 10 independent practice sessions.

c. Find the probability that she needs more than 3 arrows in exactly 3 of the sessions.

d. Find the mean and the standard deviation of the total number of arrows she shoots in the 10 sessions.

(8 marks)

$X \sim Geo(p)$

a) Within 2 arrows means $X \leq 2$

$1 - (1 - p)^2 = 0.64$

$(1 - p)^2 = 0.36$

$1 - p = 0.6$

$p = 0.4$

b) First 3 arrows all miss

$P(X > 3) = 0.6^3$

$P(X > 3) = 0.216$

c) Y = number of sessions with more than 3 arrows

$Y \sim B(10, 0.216)$

$P(Y = 3) = 10C3 \times 0.216^3 \times 0.784^7$

$P(Y = 3) = 120 \times 0.010077... \times 0.182059...$

$P(Y = 3) = 0.2202$

d) $T = X_1 + X_2 + ... + X_{10}$, independent

$E(T) = 10 \times \frac{1}{0.4} = 25$

$Var(T) = 10 \times \frac{0.6}{0.4^2}$

$Var(T) = 10 \times 3.75 = 37.5$

$sd = \sqrt{37.5}$

Mean 25, standard deviation 6.12

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11 A bag contains 10 balls, 2 of which are gold. Priya, Quinn and Rhys take turns, in that order, to draw a ball at random from the bag. The ball is replaced before the next draw. The first person to draw a gold ball wins.

a. Find the probability that Rhys wins on his 2nd turn.

b. Show that the probability that Priya wins is $\frac{25}{61}$

c. Find the probability that Rhys wins, giving your answer as a fraction.

d. Given that the game ends within the first 5 draws, find the probability that Rhys wins.

e. Find the mean and the standard deviation of the number of draws in a game.

(10 marks)

N = number of draws, $N \sim Geo(0.2)$

a) Rhys's 2nd turn is draw 6

$P(N = 6) = 0.8^5 \times 0.2$

$P(N = 6) = 0.065536$

$\boxed{0.0655}$

b) A full round of 3 misses: $0.8^3 = 0.512$

Priya wins on draw 1, 4, 7, ...

$P(\text{Priya}) = 0.2 + 0.512 \times 0.2 + 0.512^2 \times 0.2 + \ldots$

$P(\text{Priya}) = \dfrac{0.2}{1 - 0.512}$

$P(\text{Priya}) = \dfrac{0.2}{0.488}$

$\boxed{P(\text{Priya wins}) = \dfrac{25}{61}}$

c) Rhys wins on draw 3, 6, 9, ...: first term $0.8^2 \times 0.2$

$P(\text{Rhys}) = \dfrac{0.128}{0.488}$

$\boxed{P(\text{Rhys wins}) = \dfrac{16}{61}}$

d) Within 5 draws Rhys can only win on draw 3

$P(N \leq 5) = 1 - 0.8^5 = 0.67232$

$P(\text{Rhys wins} \mid N \leq 5) = \dfrac{0.128}{0.67232}$

$\boxed{0.1904}$

e) Mean and variance of $Geo(0.2)$

$E(N) = \dfrac{1}{0.2} = 5$

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$$\text{Var}(N) = \frac{0.8}{0.2^2} = 20$$

$$\text{sd} = \sqrt{20}$$

Mean 5, standard deviation 4.47

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12 At an arcade, Kayla pays £2 for each attempt at a reaction challenge. Each attempt is successful with probability 0.15, independently of all other attempts.

a. Kayla keeps paying for attempts until she is successful. Find the expected amount she spends.

b. Find the probability that she spends more than £20

Kayla decides instead that she will stop after her first success or after 8 attempts, whichever comes first. The random variable N is the number of attempts she makes.

c. Find the probability that she is successful.

d. Explain why $P(N \geq n) = 0.85^{n-1}$ for $n = 1, 2, ..., 8$

e. Given that $E(N) = P(N \geq 1) + P(N \geq 2) + ... + P(N \geq 8)$, find the expected amount she spends.

(10 marks)

a) No limit: $X \sim Geo(0.15)$

$$E(X) = \frac{1}{0.15} = 6.666...$$

$$E(cost) = 2 \times 6.666...$$

£13.33

b) More than £20 means more than 10 attempts

$$P(X > 10) = 0.85^{10}$$

0.1969

c) Fails only if all 8 attempts fail

$$P(success) = 1 - 0.85^8$$

$$P(success) = 1 - 0.272490...$$

0.7275

d) She only makes an nth attempt if she has not yet succeeded

$N \geq n \Leftrightarrow$ the first $n-1$ attempts all fail

$P(N \geq n) = 0.85^{n-1}$ for $n \leq 8$

e) Use the given formula

$$E(N) = 1 + 0.85 + 0.85^2 + ... + 0.85^7$$

Geometric series, 8 terms, $a = 1$, $r = 0.85$

$$E(N) = \frac{1 - 0.85^8}{1 - 0.85}$$

$$E(N) = \frac{0.727509...}{0.15} = 4.850...$$

$$E(cost) = 2 \times 4.850...$$

£9.70

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