# The negative binomial distribution

Worked solutions

**Further Maths topic practice**

This topic online

**1** The random variable $X \sim NB(3, 0.25)$.

a. Find $P(X = 7)$.

b. Find $E(X)$ and $Var(X)$.

(4 marks)

a) $P(X = x) = (x - 1)C(r - 1) \times p^r \times (1 - p)^{x - r}$

2 successes in the first 6 trials, then a success on the 7th

$P(X = 7) = 6C2 \times 0.25^3 \times 0.75^4$

$P(X = 7) = 15 \times 0.015625 \times 0.31640625$

$P(X = 7) = 0.074157...$

$\boxed{P(X = 7) = 0.0742}$

b) $E(X) = \frac{r}{p}$, $Var(X) = \frac{r(1 - p)}{p^2}$

$E(X) = \frac{3}{0.25} = 12$

$Var(X) = \frac{3 \times 0.75}{0.25^2}$

$Var(X) = \frac{2.25}{0.0625}$

$\boxed{E(X) = 12, Var(X) = 36}$


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**2** Zara is practising basketball free throws. The probability that she scores with any throw is 0.7

a. Find the probability that she scores for the 4th time on her 6th throw.

b. State two assumptions needed for the model you used in part (a) to be valid.

(4 marks)

a) X = throws up to and including the 4th score

$X \sim NB(4, 0.7)$

3 scores in the first 5 throws, then a score

$P(X = 6) = 5C3 \times 0.7^4 \times 0.3^2$

$P(X = 6) = 10 \times 0.2401 \times 0.09$

$P(X = 6) = 0.21609$

**0.2161**

b) Assumptions

The throws are independent of each other

**P(score) stays 0.7 on every throw (no tiring or improving)**

**3** A cereal company puts a token inside 4% of its boxes, with at most one token in a box. Anyone who collects 5 tokens can claim a free mug. Ellis buys boxes of the cereal one at a time until he can claim a mug.

a. Find the probability that he can claim a mug when he buys his 100th box.

b. Find the mean and the standard deviation of the number of boxes he must buy.

(5 marks)

$Y \sim NB(5, 0.04)$

a) 4 tokens in the first 99 boxes, then a token in box 100

$P(Y = 100) = 99C4 \times 0.04^5 \times 0.96^{95}$

$P(Y = 100) = 3764376 \times 0.04^5 \times 0.96^{95}$

$P(Y = 100) = 0.007975...$

**0.0080**

b) $E(Y) = \frac{r}{p}$, $Var(Y) = \frac{r(1-p)}{p^2}$

$E(Y) = \frac{5}{0.04} = 125$

$Var(Y) = \frac{5 \times 0.96}{0.04^2} = 3000$

$sd = \sqrt{3000}$

**Mean 125 boxes, standard deviation 54.8**


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**4** Each seed of a certain variety germinates with probability 0.6, independently of other seeds. Nadia plants seeds one at a time until 5 of them have germinated. The random variable $S$ is the number of seeds she plants.

a. Explain why $P(S \leq 8) = P(B \geq 5)$, where $B \sim B(8, 0.6)$.

b. Find $P(S \leq 8)$.

c. Find the probability that Nadia needs to plant more than 10 seeds. (7 marks)

$S \sim NB(5, 0.6)$

a) Same event, counted two ways

$S \leq 8$ means the 5th germination is within the first 8 seeds

$\Leftrightarrow$ at least 5 of the first 8 seeds germinate

Number germinating in 8 seeds is $B \sim B(8, 0.6)$, so $P(S \leq 8) = P(B \geq 5)$

b) Add the binomial terms for 5 to 8

$P(B = 5) = {}^8C_5 \times 0.6^5 \times 0.4^3 = 0.278692$

$P(B = 6) = {}^8C_6 \times 0.6^6 \times 0.4^2 = 0.209019$

$P(B = 7) = {}^8C_7 \times 0.6^7 \times 0.4 = 0.089580$

$P(B = 8) = 0.6^8 = 0.016796$

$P(S \leq 8) = 0.594086...$

$\boxed{0.5941}$

c) $S > 10 \Leftrightarrow$ at most 4 of the first 10 seeds germinate

$W \sim B(10, 0.6)$

$P(W \leq 4) = P(0) + P(1) + P(2) + P(3) + P(4)$

$P(W \leq 4) = 0.000105 + 0.001573 + 0.010617 + 0.042467 + 0.111477$

$P(W \leq 4) = 0.166238...$

$\boxed{P(S > 10) = 0.1662}$


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**5** The random variable $X$ has a negative binomial distribution with mean 12 and standard deviation 6

a. Find the parameters $r$ and $p$ of the distribution.

b. Find $P(X \leq 5)$.

(6 marks)

a) Mean $\frac{r}{p}$, variance $\frac{r(1-p)}{p^2} = 6^2$

$$\frac{r}{p} = 12$$

$$\frac{r(1-p)}{p^2} = 36$$

Divide the second by the first

$$\frac{1-p}{p} = 3$$

$$1 - p = 3p$$

$$p = \frac{1}{4}$$

$$r = 12 \times \frac{1}{4} = 3$$

$$\boxed{r = 3,\ p = \frac{1}{4}}$$

b) $X \geq 3$, so add $x = 3, 4, 5$

$$P(X = 3) = \left(\frac{1}{4}\right)^3 = \frac{8}{512}$$

$$P(X = 4) = {}^3C_2 \times \left(\frac{1}{4}\right)^3 \times \frac{3}{4} = \frac{18}{512}$$

$$P(X = 5) = {}^4C_2 \times \left(\frac{1}{4}\right)^3 \times \left(\frac{3}{4}\right)^2 = \frac{27}{512}$$

$$P(X \leq 5) = \frac{53}{512}$$

$$\boxed{P(X \leq 5) = 0.1035}$$


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**6** Mo is training to take penalty kicks. The probability that he scores with any penalty is $p$, independently of other penalties. In each training session he takes penalties until he has scored 3 times. The random variable $T$ is the number of penalties he takes in a session.

His coach wants the mean of $T$ to be at most 15 and the variance of $T$ to be less than 30

a. Find the smallest value of $p$ for which the mean of $T$ is at most 15

b. Show that the variance condition requires $10p^2 + p - 1 > 0$

c. Hence find the range of values of $p$ that satisfy both conditions.

(7 marks)

$T \sim NB(3, p)$

a) Mean at most 15

$$\frac{3}{p} \leq 15$$

$$3 \leq 15p$$

$$p \geqslant 0.2$$

**Smallest $p = 0.2$**

b) Variance less than 30

$$\frac{3(1-p)}{p^2} < 30$$

$p^2 > 0$ so multiplying keeps the inequality

$$3 - 3p < 30p^2$$

$$30p^2 + 3p - 3 > 0$$

$$10p^2 + p - 1 > 0$$

c) Critical values from the quadratic formula

$$p = \frac{-1 \pm \sqrt{41}}{20}$$

$p = 0.2702$ or $p = -0.3702$

Positive quadratic, outside the roots, and $p > 0$

$$p > 0.2702$$

Combine with $p \geqslant 0.2$

$$0.270 < p \leq 1$$


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**7** A warehouse robot picks items from shelves. Each attempt to grip an item is successful with probability 0.3, independently of all other attempts, and the robot keeps trying until it grips the item. The robot picks 3 items, one after another. $X_1$, $X_2$ and $X_3$ are the numbers of attempts for the 1st, 2nd and 3rd items, and $Y = X_1 + X_2 + X_3$

a. State, with a reason, the distribution of $Y$.

b. Find $P(Y = 6)$.

c. Find $E(Y)$ and $Var(Y)$.

d. Find $P(Y = 6 \mid X_1 = 2)$.

(8 marks)

a) Each $X_i \sim Geo(0.3)$, independent

Y counts the attempts up to and including the 3rd success

$$Y \sim NB(3, 0.3)$$

b) 2 successes in the first 5 attempts, then a success

$$P(Y = 6) = {}^5C_2 \times 0.3^3 \times 0.7^3$$

$$P(Y = 6) = 10 \times 0.027 \times 0.343$$

$$P(Y = 6) = 0.09261$$

$$0.0926$$

c) $E(Y) = \frac{r}{p}$, $Var(Y) = \frac{r(1-p)}{p^2}$

$$E(Y) = \frac{3}{0.3} = 10$$

$$Var(Y) = \frac{3 \times 0.7}{0.09} = \frac{70}{3}$$

$$E(Y) = 10, \ Var(Y) = 23.3$$

d) $X_1 = 2$ uses 2 attempts, so need $X_2 + X_3 = 4$

$X_2 + X_3 \sim NB(2, 0.3)$, independent of $X_1$

$$P(X_2 + X_3 = 4) = {}^3C_1 \times 0.3^2 \times 0.7^2$$

$$P(X_2 + X_3 = 4) = 3 \times 0.09 \times 0.49$$

$$0.1323$$


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**8** At a sports day, Tomasz must choose one of two challenges.

- Challenge A: he takes basketball shots, and he wins if he scores for the 3rd time within his first 5 shots. The probability that he scores with any shot is 0.45
- Challenge B: he throws darts, and he wins if he hits the bull for the first time within his first 3 throws. The probability that he hits the bull with any throw is 0.25

All shots and throws are independent. Showing your calculations clearly, determine which challenge gives Tomasz the greater probability of winning.

(7 marks)

Challenge A: $X \sim NB(3, 0.45)$, want $P(X \leq 5)$

$P(X = 3) = 0.45^3 = 0.091125$

$P(X = 4) = {}^3C_2 \times 0.45^3 \times 0.55 = 0.150356$

$P(X = 5) = {}^4C_2 \times 0.45^3 \times 0.55^2 = 0.165392$

$P(X \leq 5) = 0.406873...$

$P(\text{win A}) = 0.4069$

Challenge B: $Y \sim Geo(0.25)$, want $P(Y \leq 3)$

$P(Y \leq 3) = 1 - 0.75^3$

$P(Y \leq 3) = 0.578125$

$P(\text{win B}) = 0.5781$

Compare

$0.5781 > 0.4069$

Challenge B gives the greater probability of winning

**9** Items on a production line are checked one at a time. Each item is defective with probability 0.1, independently of other items.

a. Find the probability that the 3rd defective item found is the 15th item checked.

b. Find the probability that exactly 2 of the first 10 items checked are defective.

c. Given that the 3rd defective item found is the 15th item checked, show that the probability that the first two defective items were both among the first 7 items checked is $\frac{3}{13}$

(8 marks)

a) $X$ = item number of the 3rd defective, $X \sim NB(3, 0.1)$

$P(X = 15) = {}^{14}C_2 \times 0.1^3 \times 0.9^{12}$

$P(X = 15) = 91 \times 0.001 \times 0.282429...$

$\boxed{0.0257}$

b) $0$ = defectives in the first 10, $0 \sim B(10, 0.1)$

$P(0 = 2) = {}^{10}C_2 \times 0.1^2 \times 0.9^8$

$P(0 = 2) = 45 \times 0.01 \times 0.430467...$

$\boxed{0.1937}$

c) 2 defectives in items 1 to 7, none in 8 to 14, item 15 defective

$P(\text{both events}) = {}^{7}C_2 \times 0.1^2 \times 0.9^5 \times 0.9^7 \times 0.1$

$P(\text{both events}) = 21 \times 0.1^3 \times 0.9^{12}$

Divide by $P(X = 15)$

$$P(\text{first two in 1 to 7} \mid X = 15) = \frac{21 \times 0.1^3 \times 0.9^{12}}{91 \times 0.1^3 \times 0.9^{12}}$$

$$P(\text{first two in 1 to 7} \mid X = 15) = \frac{21}{91}$$

$$\boxed{= \frac{3}{13}}$$


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**10** At a school fair, a stall charges £1 per go on a game. On each go the probability of winning a token is 0.2, independently of other goes. Ifeoma plays until she has won 2 tokens, and then she exchanges the 2 tokens for a prize worth £8

a. Find Ifeoma's expected profit.

b. Find the probability that Ifeoma makes a profit.

c. Find the probability that Ifeoma makes a loss of more than £5

d. Find the prize value that would make the game fair, so that the expected profit is zero.

(8 marks)

$X$ = number of goes, $X \sim NB(2, 0.2)$

a) Profit $= 8 - X$

$$E(X) = \frac{2}{0.2} = 10$$

$E(\text{profit}) = 8 - 10$

**-£2 (expected loss of £2)**

b) Profit: $8 - X > 0$ so $X \leqslant 7$

$X \leqslant 7 \Leftrightarrow$ at least 2 tokens in the first 7 goes

$W \sim B(7, 0.2)$

$P(W \geqslant 2) = 1 - P(W = 0) - P(W = 1)$

$P(W \geqslant 2) = 1 - 0.8^7 - 7 \times 0.2 \times 0.8^6$

$P(W \geqslant 2) = 1 - 0.209715 - 0.367002$

**0.4233**

c) Loss more than £5: $X - 8 > 5$ so $X \geqslant 14$

$X \geqslant 14 \Leftrightarrow$ at most 1 token in the first 13 goes

$V \sim B(13, 0.2)$

$P(V \leqslant 1) = 0.8^{13} + 13 \times 0.2 \times 0.8^{12}$

$P(V \leqslant 1) = 0.054976 + 0.178671$

**0.2336**

d) Prize £$k$, fair when $E(\text{profit}) = 0$

$k - 10 = 0$

**Prize worth £10**


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**11** The random variable $X \sim \text{NB}(4, 0.3)$.

a. Show that, for $x \geq 4$, $\dfrac{\text{P}(X = x + 1)}{\text{P}(X = x)} = \dfrac{0.7x}{x - 3}$

b. Hence find the most likely value or values of $X$.

c. Find the probability that $X$ takes a most likely value.

(8 marks)

a) Write out both probabilities

$$P(X = x) = (x-1)C3 \times 0.3^4 \times 0.7^{x-4}$$

$$P(X = x+1) = xC3 \times 0.3^4 \times 0.7^{x-3}$$

$$\text{Ratio} = \frac{xC3}{(x-1)C3} \times 0.7$$

$$\frac{xC3}{(x-1)C3} = \frac{x(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\frac{xC3}{(x-1)C3} = \frac{x}{x-3}$$

$$\boxed{\text{Ratio} = \frac{0.7x}{x-3}}$$

b) Probabilities increase while the ratio $> 1$

$$\frac{0.7x}{x-3} > 1$$

$$0.7x > x - 3$$

$$x < 10$$

At $x = 10$ the ratio is $\frac{7}{7} = 1$, so $P(X = 11) = P(X = 10)$

Increasing up to $x = 10$, equal at 10 and 11, then decreasing

$$\boxed{\text{Most likely values: 10 and 11}}$$

c) Both have the same probability

$$P(X = 10) = 9C3 \times 0.3^4 \times 0.7^6$$

$$P(X = 10) = 84 \times 0.0081 \times 0.117649$$

$$P(X = 10) = 0.080048...$$

$$\boxed{P(X = 10) = P(X = 11) = 0.0800}$$


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**12** Tickets for a charity tombola cost £2 each. Each ticket wins a prize with probability 0.25, independently of other tickets.

Leila buys tickets one at a time until she has won $r$ prizes, and then she stops. The charity gives her £1 for her 1st prize, £2 for her 2nd prize, and in general £$w$ for her $w$th prize.

a. Show that Leila's expected profit is £$\frac{1}{2}r(r - 15)$

b. Find the smallest value of $r$ for which Leila's expected profit is positive.

c. Given that $r = 4$, find the standard deviation of Leila's profit.

Kofi buys exactly 12 tickets.

d. Explain why the probability that Kofi wins at least 3 prizes is equal to the probability that Leila, with $r = 3$, needs at most 12 tickets, and find this probability.

(11 marks)

$X$ = tickets Leila buys, $X \sim NB(r, 0.25)$

a) Prize money is fixed: $1 + 2 + \ldots + r$

$$\text{Prizes} = \frac{r(r+1)}{2}$$

$$E(X) = \frac{r}{0.25} = 4r$$

$$E(\text{cost}) = 2 \times 4r = 8r$$

$$E(\text{profit}) = \frac{r(r+1)}{2} - 8r$$

$$E(\text{profit}) = \frac{r^2 + r - 16r}{2}$$

$$\boxed{E(\text{profit}) = \frac{1}{2}r(r - 15)}$$

b) Need $\frac{1}{2}r(r - 15) > 0$ with $r > 0$

$r > 15$

$\boxed{r = 16}$

c) $r = 4$: prizes $1 + 2 + 3 + 4 = 10$

$\text{Profit} = 10 - 2X$

$$\text{Var}(X) = \frac{4 \times 0.75}{0.25^2} = 48$$

$$\text{Var}(\text{profit}) = 2^2 \times 48 = 192$$

$$\text{sd} = \sqrt{192}$$

$\boxed{£13.86}$

d) Leila needs at most 12 tickets ⇔ at least 3 wins in the first 12


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That is the same as 12 tickets giving at least 3 prizes

$K \sim B(12, 0.25)$

$P(K \geq 3) = 1 - P(K = 0) - P(K = 1) - P(K = 2)$

$P(K \geq 3) = 1 - 0.031676 - 0.126705 - 0.232293$

$\boxed{0.6093}$


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