Type I and type II errors

Worked solutions

Further Maths topic practice

This topic online

1 A factory has found that 6% of the phone cases it makes are faulty. After a new moulding machine is installed, the manager, Gareth, tests whether the proportion of faulty cases has increased, using H_0: $p = 0.06$ and H_1: $p > 0.06$

a. Describe, in context, what is meant by a Type I error in this test.

b. Describe, in context, what is meant by a Type II error in this test.

c. Gareth's test does not reject H_0. State which type of error he could have made.

(3 marks)

Type I: reject H_0 when H_0 is true

a) Concludes the proportion of faulty cases has increased when it is still 0.06

Type II: do not reject H_0 when H_0 is false

b) Concludes there is no increase when the proportion really has increased

c) H_0 was not rejected, so a Type I error is impossible

Type II error

2 A seed company claims that 35% of its sweet pea seeds produce white flowers. Hannah thinks the proportion is lower. She plants 20 seeds and tests H_0: $p = 0.35$ against H_1: $p < 0.35$ at the 5% level of significance. The critical region is $X \leq 3$, where X is the number of seeds that produce white flowers.

a. Find P(Type I error) for this test.

b. Explain why P(Type I error) is not equal to 0.05

(3 marks)

a) Type I error: X in the critical region when H_0 is true

$X \sim B(20, 0.35)$

$P(\text{Type I}) = P(X \leq 3)$

$P(\text{Type I}) = 0.044375...$

0.0444

b) X is discrete

$P(X \leq 4) = 0.1182$

$0.1182 > 0.05$, so $X \leq 3$ is the largest region allowed

The probabilities go up in steps, so none is exactly 0.05

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3 The number of calls to a helpline between 9 am and 10 am has a Poisson distribution with mean 4. After an advert is shown on television, the manager wants to test whether the mean number of calls in this hour has increased.

a. Using a 5% level of significance, find the critical region for the test. State your hypotheses clearly.

b. Write down P(Type I error) for this test.

(4 marks)

a) λ = mean number of calls from 9 am to 10 am

$H_0: \lambda = 4$

$H_1: \lambda > 4$

Under H_0, $X \sim Po(4)$

$P(X \geqslant 8) = 1 - P(X \leqslant 7) = 0.0511$

more than 0.05

$P(X \geqslant 9) = 1 - P(X \leqslant 8) = 0.0214$

less than 0.05

$X \geqslant 9$

b) P(Type I) = P(in the critical region | H_0 true)

0.0214

4 Past records show that 20% of visitors to a museum buy a guidebook. The museum redesigns the guidebook and wants to test whether this proportion has increased. A random sample of 25 visitors is taken and the number who buy a guidebook, X, is recorded.

a. Find the critical region for a test of H_0: $p = 0.2$ against H_1: $p > 0.2$ at the 5% level of significance.

b. Given that the true proportion is now 0.3, find P(Type II error).

(5 marks)

a) Under H_0, $X \sim B(25, 0.2)$

$P(X \geqslant 8) = 1 - P(X \leqslant 7) = 0.1091$

more than 0.05

$P(X \geqslant 9) = 1 - P(X \leqslant 8) = 0.0468$

less than 0.05

$X \geqslant 9$

b) Type II: do not reject H_0 when $p = 0.3$

$X \sim B(25, 0.3)$

$P(\text{Type II}) = P(X \leqslant 8)$

$P(\text{Type II}) = 0.676928...$

0.6769

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5 A school receives emails from parents at a mean rate of 8 per day. After a new website is launched, the head teacher wants to test, at the 5% level of significance, whether the mean number of emails per day has changed, using the number of emails received on one randomly chosen day.

a. Find the critical region for the test. The probability in each tail should be less than 0.025 and as close to 0.025 as possible. State the probability in each tail.

b. Find P(Type I error) for this test.

c. Given that the mean number of emails per day is in fact 12, find P(Type II error). (7 marks)

a) $H_0: \lambda = 8$, $H_1: \lambda \neq 8$

Under H_0, $X \sim Po(8)$

Lower tail, at most 0.025

$P(X \leqslant 2) = 0.0138$

$P(X \leqslant 3) = 0.0424$

Upper tail, at most 0.025

$P(X \geqslant 14) = 1 - P(X \leqslant 13) = 0.0342$

$P(X \geqslant 15) = 1 - P(X \leqslant 14) = 0.0173$

$X \leqslant 2$ (0.0138) or $X \geqslant 15$ (0.0173)

b) Add the two tails

$P(\text{Type I}) = 0.013754... + 0.017257...$

$P(\text{Type I}) = 0.031011...$

0.0310

c) Type II: X not in the critical region when $\lambda = 12$

$X \sim Po(12)$

$P(\text{Type II}) = P(3 \leqslant X \leqslant 14)$

$P(\text{Type II}) = P(X \leqslant 14) - P(X \leqslant 2)$

$P(\text{Type II}) = 0.772025... - 0.000522...$

0.7715

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6 A game developer states that the probability of finding a rare item in a loot box is 0.2. Zara believes the probability is lower. She opens loot boxes one at a time until she finds a rare item, and records the number of boxes opened, X, including the one containing the rare item.

a. Using a 5% level of significance, find the critical region for a test of H_0: $p = 0.2$ against H_1: $p < 0.2$

b. Write down P(Type I error).

c. Given that the probability of finding a rare item is actually 0.1, find P(Type II error). (6 marks)

a) Under H_0, $X \sim Geo(0.2)$

p smaller means more boxes: upper tail

$$P(X \geqslant c) = 0.8^{c-1}$$

$$0.8^{c-1} \leqslant 0.05$$

$$c - 1 \geqslant \frac{\ln 0.05}{\ln 0.8} = 13.43$$

$$c \geqslant 14.43$$

Check: $0.8^{13} = 0.0550$, $0.8^{14} = 0.0440$

$$\boxed{X \geqslant 15}$$

b)

$$P(\text{Type I}) = P(X \geqslant 15) = 0.8^{14}$$

$$\boxed{0.0440}$$

c) Type II: $X \leqslant 14$ when $p = 0.1$

$$P(X \leqslant 14) = 1 - 0.9^{14}$$

$$P(X \leqslant 14) = 1 - 0.228768...$$

$$\boxed{0.7712}$$

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7 Cyclists pass an automatic counter on a cycle path at a mean rate of 3 every 10 minutes. Roadworks start nearby, and in a randomly chosen 40-minute period the counter records 6 cyclists.

a. Test, at the 5% level of significance, whether there is evidence of a decrease in the rate at which cyclists pass the counter. State your hypotheses clearly.

A further randomly chosen 20-minute period is observed and the test is repeated, again at the 5% level.

b. Find the critical region for this test.

c. Given that the true rate is now 1.5 cyclists every 10 minutes, find the probability of a Type II error. (9 marks)

a) Scale to 40 minutes: $4 \times 3 = 12$

$H_0: \lambda = 12$

$H_1: \lambda < 12$

Under H_0, $X \sim Po(12)$

$P(X \leq 6) = 0.0458$

$0.0458 < 0.05$, so reject H_0

Evidence that the rate of cyclists has decreased

b) 20 minutes: $\lambda = 2 \times 3 = 6$

Under H_0, $X \sim Po(6)$

$P(X \leq 1) = 7e^{-6} = 0.0174$

$P(X \leq 2) = 25e^{-6} = 0.0620$

$0.0620 > 0.05$, so stop at 1

$X \leq 1$

c) New rate for 20 minutes: $2 \times 1.5 = 3$

$X \sim Po(3)$

Type II: not in the critical region

$P(X \geqslant 2) = 1 - P(X \leq 1)$

$P(X \geqslant 2) = 1 - 4e^{-3}$

$P(X \geqslant 2) = 0.800851...$

0.8009

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8 The mass of honey in a jar is normally distributed with mean μ grams and standard deviation 8 grams. The label says that μ = 250. A trading standards officer suspects that the jars are underfilled. She weighs a random sample of 16 jars and calculates the sample mean mass $\bar{X}$.

a. Using a 5% level of significance, find the critical region for $\bar{X}$. State your hypotheses clearly.

b. Given that μ is actually 245, find P(Type II error). (7 marks)

a)

$H_0: \mu = 250$

$H_1: \mu < 250$

Under H_0, $X\text{-bar} \sim N\left(250, \frac{8^2}{16}\right)$

$sd = \frac{8}{\sqrt{16}} = 2$

Lower 5% point: $z = -1.6449$

$c = 250 - 1.6449 \times 2$

$c = 246.7103...$

$X\text{-bar} < 246.71$

b) Type II: not in the critical region when $\mu = 245$

$X\text{-bar} \sim N(245, 2^2)$

$P(X\text{-bar} \geq 246.7103) = P\left(Z \geq \frac{(246.7103 - 245)}{2}\right)$

$P(X\text{-bar} \geq 246.7103) = P(Z \geq 0.8551)$

0.1962

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9 A machine makes bolts whose lengths are normally distributed with mean μ mm and standard deviation 0.5 mm. To check that μ = 12, an engineer measures a random sample of 20 bolts and rejects H_0: μ = 12 in favour of H_1: μ ≠ 12 if the sample mean $\bar{X}$ satisfies $\bar{X} < 11.8$ or $\bar{X} > 12.2$

a. Find P(Type I error) for this test.

b. Given that μ = 12.3, find P(Type II error).

(6 marks)

a) Under H_0

$$\text{X-bar} \sim N\left(12, \frac{0.5^2}{20}\right)$$

$$\text{sd} = \frac{0.5}{\sqrt{20}} = 0.111803...$$

Type I: X-bar in the critical region when $\mu = 12$

$$P(\text{X-bar} < 11.8) = P\left(Z < \frac{-0.2}{0.111803}\right) = P(Z < -1.7889)$$

Symmetrical, so double it

$$P(\text{Type I}) = 2 \times 0.036819...$$

$$\boxed{0.0736}$$

b) Type II: X-bar not in the critical region when $\mu = 12.3$

$$\text{X-bar} \sim N(12.3, 0.111803^2)$$

$$P(11.8 \leq \text{X-bar} \leq 12.2) = P\left(\frac{(11.8 - 12.3)}{0.111803} \leq Z \leq \frac{(12.2 - 12.3)}{0.111803}\right)$$

$$P(11.8 \leq \text{X-bar} \leq 12.2) = P(-4.4721 \leq Z \leq -0.8944)$$

$$P(11.8 \leq \text{X-bar} \leq 12.2) = 0.185547... - 0.000004...$$

$$\boxed{0.1855}$$

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10 A council receives complaints about missed bin collections at a mean rate of 4.5 per month. After a change of contractor, the council records the number of complaints in a series of 2-month periods and uses each period to test whether the mean number of complaints per month has changed.

a. Stating your hypotheses clearly and using a 5% level of significance, find the critical region for this test. The probability in each tail should be less than 0.025 and as close to 0.025 as possible.

b. State P(Type I error) for this test.

c. The test is carried out for each of the 6 two-month periods in a year. Given that the mean rate is in fact still 4.5 per month, find the probability that at least 2 of these tests give a significant result.

d. Given instead that the mean rate is actually 3 per month, find P(Type II error) for one test.

(10 marks)

a) Scale to 2 months: $2 \times 4.5 = 9$

$H_0: \lambda = 9$

$H_1: \lambda \neq 9$

Under H_0, $X \sim Po(9)$

Lower tail

$P(X \leqslant 3) = 0.0212$

$P(X \leqslant 4) = 0.0550$

Upper tail

$P(X \geqslant 15) = 1 - P(X \leqslant 14) = 0.0415$

$P(X \geqslant 16) = 1 - P(X \leqslant 15) = 0.0220$

$\boxed{X \leqslant 3 \text{ or } X \geqslant 16}$

b) Add the tails

$P(\text{Type I}) = 0.021226... + 0.022036...$

$\boxed{0.0433}$

c) Each test significant with probability 0.043262...

$S \sim B(6, 0.043262)$

$P(S \geqslant 2) = 1 - P(S = 0) - P(S = 1)$

$P(S \geqslant 2) = 1 - 0.956738^6 - 6(0.043262)(0.956738)^5$

$P(S \geqslant 2) = 0.024989...$

$\boxed{0.0250}$

d) 2 months at 3 per month: $\lambda = 6$

$X \sim Po(6)$

Type II: $4 \leqslant X \leqslant 15$

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$P(4 \leq X \leq 15) = P(X \leq 15) - P(X \leq 3)$

$P(4 \leq X \leq 15) = 0.999491... - 0.151204...$

0.8483

11 The number of weeds in a 1 m^2 quadrat in a playing field has a Poisson distribution with mean 3. A groundsman uses a new treatment and believes it has reduced the mean number of weeds. He counts the weeds in 50 randomly placed quadrats and calculates the sample mean $\bar{X}$.

a. Using a suitable approximation and a 5% level of significance, find the critical region for $\bar{X}$ for a test of H_0: $\lambda = 3$ against H_1: $\lambda < 3$

b. Given that the mean number of weeds per quadrat is in fact 2.6, find P(Type II error).

c. The groundsman uses a 1% level of significance instead, with the same sample size. Without further calculation, state, with a reason, the effect on P(Type II error).

(9 marks)

a) CLT: Poisson mean = variance = 3, $n = 50$

$\text{X-bar} \approx N(3, \frac{3}{50})$

$sd = \sqrt{0.06} = 0.244949...$

Lower 5% point: $z = -1.6449$

$c = 3 - 1.6449 \times 0.244949$

$c = 2.59709...$

X-bar < 2.597

b) Type II: X-bar $\geqslant c$ when $\lambda = 2.6$

$\text{X-bar} \approx N(2.6, \frac{2.6}{50})$

$sd = \sqrt{0.052} = 0.228035...$

$P(\text{X-bar} \geqslant 2.59709) = P(Z \geqslant \frac{(2.59709 - 2.6)}{0.228035})$

$P(\text{X-bar} \geqslant 2.59709) = P(Z \geqslant -0.0128)$

0.5051

c) 1% level: c moves further below 3

The critical region is smaller

P(Type II error) increases: H_0 is less likely to be rejected when it is false

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12 A café owner, Mei, knows that 25% of her customers used to pay with cash. She wants to test whether this proportion has changed. She records how 30 randomly chosen customers pay.

a. Stating your hypotheses clearly, find the critical region for a test at the 5% level of significance. The probability in each tail should be less than 0.025 and as close to 0.025 as possible.

b. Find P(Type I error) for this test.

c. Given that the true proportion paying with cash is 0.45, find P(Type II error).

d. In Mei's sample, 13 customers pay with cash. State the conclusion of the test and state, with a reason, which type of error Mei might have made.

(10 marks)

a)

$H_0: p = 0.25$

$H_1: p \neq 0.25$

Under H_0, $X \sim B(30, 0.25)$

Lower tail

$P(X \leqslant 2) = 0.0106$

$P(X \leqslant 3) = 0.0374$

Upper tail

$P(X \geqslant 12) = 1 - P(X \leqslant 11) = 0.0507$

$P(X \geqslant 13) = 1 - P(X \leqslant 12) = 0.0216$

$X \leqslant 2$ or $X \geqslant 13$

b) Add the tails

$P(\text{Type I}) = 0.010596... + 0.021594...$

0.0322

c) Type II: $3 \leqslant X \leqslant 12$ when $p = 0.45$

$X \sim B(30, 0.45)$

$P(3 \leqslant X \leqslant 12) = P(X \leqslant 12) - P(X \leqslant 2)$

$P(3 \leqslant X \leqslant 12) = 0.359178... - 0.000005...$

0.3592

d) 13 is in the critical region

Reject H_0: evidence the proportion paying with cash has changed

H_0 rejected: it might really be true

Possibly a Type I error

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