



- 1 Prove algebraically that the sum of an odd number and an even number is always odd.

(2 marks)

Even number: $2n$

Odd number: $2m + 1$ (n, m integers)

$$\begin{aligned} 2n + (2m + 1) &= 2n + 2m + 1 \\ &= 2(n + m) + 1 \end{aligned}$$

$2(n + m)$ is a multiple of 2, so it is even; one more makes it odd

Sum is always odd

- 2 Sam says, " $n^2 + n + 11$ is a prime number for every positive integer n ."

Show that Sam is wrong.

(2 marks)

Look for a counter-example

$$n = 1: 1 + 1 + 11 = 13 \text{ (prime)}$$

$$n = 10: 100 + 10 + 11 = 121$$

$$121 = 11 \times 11$$

$n = 10$ gives 121, not prime, so Sam is wrong

- 3 Prove that $n^2 + n$ is even for every integer n .

(2 marks)

Factorise

$$n^2 + n = n(n + 1)$$

n and $n + 1$ are consecutive integers, so one of them must be even

The product of an even number with any integer is even

Therefore $n^2 + n$ is even for every integer n

4 Prove algebraically that the sum of any three consecutive integers is a multiple of 3.

(3 marks)

Let the three consecutive integers be $n - 1$, n and $n + 1$, where n is an integer

$$\text{Sum} = (n - 1) + n + (n + 1) = 3n$$

$3n$ is 3 multiplied by an integer, so it is a multiple of 3

Therefore the sum of any three consecutive integers is a multiple of 3

5 Prove algebraically that the sum of the squares of any two consecutive even numbers is a multiple of 4.

(3 marks)

Let the two consecutive even numbers be $2n$ and $2n + 2$, where n is an integer

$$(2n)^2 + (2n + 2)^2 = 4n^2 + 4n^2 + 8n + 4 = 8n^2 + 8n + 4$$

Factorise

$$8n^2 + 8n + 4 = 4(2n^2 + 2n + 1)$$

$2n^2 + 2n + 1$ is an integer, so the sum is 4 multiplied by an integer: a multiple of 4

6 Prove algebraically that the product of any two consecutive odd numbers is always 1 less than a multiple of 4.

(3 marks)

Let the two consecutive odd numbers be $2n - 1$ and $2n + 1$, where n is an integer

Product = $(2n - 1)(2n + 1) = 4n^2 + 2n - 2n - 1 = 4n^2 - 1$. The middle terms cancel; this is the difference-of-two-squares identity

$4n^2$ is a multiple of 4, since n^2 is an integer

So the product is 1 less than a multiple of 4

7

Prove algebraically that the sum of the squares of any two consecutive integers is an odd number.

(4 marks)

Let the two consecutive integers be n and $n + 1$, where n is an integer

$$n^2 + (n + 1)^2 = n^2 + n^2 + 2n + 1 = 2n^2 + 2n + 1$$

Factorise the first two terms

$$2n^2 + 2n + 1 = 2(n^2 + n) + 1$$

$n^2 + n$ is an integer, so this is 2 multiplied by an integer plus 1: an odd number

Therefore the sum of the squares of any two consecutive integers is odd

8

The n th triangular number is $\frac{n(n+1)}{2}$.

Prove that the sum of the n th and the $(n + 1)$ th triangular numbers is always a square number.

(3 marks)

$$\text{sum} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}$$

Take out the common factor $(n + 1)$

$$= \frac{(n+1)(n + n + 2)}{2}$$

$$= \frac{(n+1)(2n + 2)}{2}$$

$$= (n + 1)(n + 1)$$

$$= (n + 1)^2, \text{ a square number}$$

9

The n th triangular number is $\frac{n(n+1)}{2}$.

Prove that eight times any triangular number, plus 1, is always the square of an odd number.

(4 marks)

$$8 \times \frac{n(n+1)}{2} + 1 = 4n(n+1) + 1$$

$$= 4n^2 + 4n + 1$$

$$= (2n + 1)(2n + 1)$$

$$= (2n + 1)^2$$

$2n$ is even, so $2n + 1$ is odd

$$\text{Square of the odd number } 2n + 1$$