



1 $f(x) = x^3 - 4x - 3$

Show that the equation $f(x) = 0$ has a root between $x = 2$ and $x = 3$.

(2 marks)

$$f(2) = 8 - 8 - 3 = -3$$

$$f(3) = 27 - 12 - 3 = 12$$

Sign change and f is continuous

A root lies between $x = 2$ and $x = 3$

2 The number of members of a gym is modelled by the rule

$$u_{n+1} = 1.2u_n - 50$$

where u_n is the number of members after n years.

There are 500 members now, so $u_0 = 500$.

Work out the number of members the model predicts after 2 years.

(2 marks)

u_0 is the starting value. The subscript $n + 1$ means the next year: put the current value u_n into the rule to get the next value. Repeat the rule twice for two years

$$u_1 = 1.2 \times 500 - 50 = 600 - 50 = 550$$

$$u_2 = 1.2 \times 550 - 50 = 660 - 50 = 610$$

610

3 A ball is dropped from a height of 5 m onto a hard floor.

After each bounce the ball rises to 60% of the height of the previous bounce, so

$$h_{n+1} = 0.6h_n, \text{ where } h_0 = 5$$

a. Work out the height the ball rises to after the third bounce.

b. After which bounce does the ball first rise to less than 1 m?

(3 marks)

$$a) h_1 = 0.6 \times 5 = 3, h_2 = 0.6 \times 3 = 1.8, h_3 = 0.6 \times 1.8 = 1.08 \text{ m}$$

b) $h_4 = 0.6 \times 1.08 = 0.648 \text{ m}$, which is the first height below 1 m, so the 4th bounce

4

Amara opens a savings account with £1000.

The account pays 3% interest at the end of each year, and Amara then pays in another £200.

The amount in the account after n years is given by

$$a_{n+1} = 1.03a_n + 200, \text{ where } a_0 = 1000$$

Work out the amount in the account after 3 years. Give your answer to the nearest penny.

(3 marks)

$$a_1 = 1.03 \times 1000 + 200 = 1230$$

$$a_2 = 1.03 \times 1230 + 200 = 1466.90$$

$$a_3 = 1.03 \times 1466.90 + 200 = 1710.907$$

To the nearest penny this is £1710.91

£1710.91

5

A sequence is given by the iteration formula

$$x_{n+1} = \sqrt[3]{(5x_n + 2)}, \text{ with } x_1 = 2$$

a. Work out the values of x_2 , x_3 and x_4 . Give each value correct to 3 decimal places.

b. The values of x_n approach a root of an equation. Write down this equation in the form $x^3 + ax + b = 0$

(4 marks)

Keep the full calculator value for each next iteration (use the Ans key). Round only the values you write down, otherwise rounding errors can build up

$$a) x_2 = \text{cube root of } (5 \times 2 + 2) = \text{cube root of } 12 = 2.289428... = 2.289 \text{ (3 dp)}$$

$$a) x_3 = \text{cube root of } (5 \times 2.289428 + 2) = 2.377990... = 2.378 \text{ (3 dp)}$$

$$a) x_4 = \text{cube root of } (5 \times 2.377990 + 2) = 2.403810... = 2.404 \text{ (3 dp)}$$

b) At the limit $x = \text{cube root of } (5x + 2)$, so $x^3 = 5x + 2$, which is $x^3 - 5x - 2 = 0$

6

A sequence is defined by

$$u_{n+1} = ku_n + 3$$

where k is a constant.

$$u_1 = 5 \text{ and } u_2 = 13$$

- Find the value of k
- Find the value of u_4

(4 marks)

$$a) u_2 = k \times u_1 + 3, \text{ so } 13 = 5k + 3, \text{ giving } 5k = 10 \text{ and } k = 2$$

$$b) u_3 = 2 \times 13 + 3 = 29$$

$$b) u_4 = 2 \times 29 + 3 = 61$$

7

The number of deer in a woodland is modelled by

$$p_{n+1} = 0.9p_n + 40$$

where p_n is the number of deer n years from now.

There are 300 deer now, so $p_0 = 300$.

- Work out the number of deer the model predicts in 2 years' time.
- The model predicts that in the long term the number of deer approaches a fixed value L . Work out the value of L .

(4 marks)

$$a) p_1 = 0.9 \times 300 + 40 = 310$$

$$a) p_2 = 0.9 \times 310 + 40 = 319$$

$$b) \text{ At the fixed value, } L = 0.9L + 40$$

$$b) \text{ So } 0.1L = 40, \text{ giving } L = 400$$

8

a. Show that the equation $x^3 - 7x + 3 = 0$ can be rearranged to give

$$x = \sqrt[3]{7x - 3}$$

b. Using the iteration formula $x_{n+1} = \sqrt[3]{7x_n - 3}$ with $x_1 = 2$, work out the values of x_2 , x_3 and x_4 .

Give each value correct to 3 decimal places.

(4 marks)

Keep the full calculator value for each next iteration (use the Ans key). Round only the values you write down, otherwise rounding errors can build up

a) Add $7x$ and subtract 3

$$x^3 = 7x - 3$$

a) Take the cube root of both sides

$$x = \text{cube root of } (7x - 3), \text{ as required}$$

$$\text{b) } x_2 = \text{cube root of } (7 \times 2 - 3) = \text{cube root of } 11 = 2.223980... = 2.224 \text{ (3 dp)}$$

$$\text{b) } x_3 = \text{cube root of } (7 \times 2.223980 - 3) = 2.324987... = 2.325 \text{ (3 dp)}$$

$$\text{b) } x_4 = \text{cube root of } (7 \times 2.324987 - 3) = 2.367793... = 2.368 \text{ (3 dp)}$$

9

$$f(x) = x^2 - 6x + 2$$

a. Show that the equation $f(x) = 0$ has a root between $x = 0$ and $x = 1$

b. Show that the equation $x^2 - 6x + 2 = 0$ can be rearranged to give

$$x = \frac{x^2 + 2}{6}$$

(4 marks)

This polynomial has a continuous graph: it has no jumps or breaks. If its value changes from positive to negative, the graph must pass through zero somewhere in between

$$\text{a) } f(0) = 0 - 0 + 2 = 2, \text{ which is positive}$$

$$\text{a) } f(1) = 1 - 6 + 2 = -3, \text{ which is negative}$$

a) f is continuous and changes sign between 0 and 1, so there is a root between $x = 0$ and $x = 1$

$$\text{b) Rearrange } x^2 - 6x + 2 = 0 \text{ to } 6x = x^2 + 2$$

b) Divide both sides by 6

$$x = \frac{x^2 + 2}{6}, \text{ as required}$$

10

$$f(x) = x^3 + 4x - 9$$

- Show that the equation $f(x) = 0$ has a root between $x = 1$ and $x = 2$
- Show that the equation $x^3 + 4x - 9 = 0$ can be rearranged to give

$$x = \sqrt[3]{9 - 4x}$$
- Starting with $x_0 = 1.5$, use the iteration formula $x_{n+1} = \sqrt[3]{9 - 4x_n}$ to work out the values of x_1 , x_2 and x_3 . Give each value correct to 4 decimal places.

(6 marks)

This polynomial has a continuous graph: it has no jumps or breaks. If its value changes from positive to negative, the graph must pass through zero somewhere in between

Keep the full calculator value for each next iteration (use the Ans key). Round only the values you write down, otherwise rounding errors can build up

$$a) f(1) = 1 + 4 - 9 = -4, \text{ which is negative}$$

$$a) f(2) = 8 + 8 - 9 = 7, \text{ which is positive}$$

a) f is continuous and changes sign, so there is a root between $x = 1$ and $x = 2$

b) Rearrange to $x^3 = 9 - 4x$, then take the cube root: $x = \text{cube root of } (9 - 4x)$, as required

$$c) x_1 = \text{cube root of } (9 - 4 \times 1.5) = \text{cube root of } 3 = 1.442250... = 1.4423 \text{ (4 dp)}$$

$$c) x_2 = \text{cube root of } (9 - 4 \times 1.442250) = 1.478356... = 1.4784 \text{ (4 dp)}$$

$$c) x_3 = \text{cube root of } (9 - 4 \times 1.478356) = 1.455992... = 1.4560 \text{ (4 dp)}$$

11 $f(x) = x^2 + 2x - 5$

a. Show that the equation $f(x) = 0$ has a root between $x = 1$ and $x = 2$

b. Show that the equation $x^2 + 2x - 5 = 0$ can be rearranged to give

$$x = \frac{5}{x+2}$$

c. Starting with $x_0 = 1.5$, use the iteration formula $x_{n+1} = \frac{5}{x_n + 2}$ to work out the values of x_1, x_2 and x_3 . Give each value correct to 4 decimal places.

d. Write down an estimate of this root of $f(x) = 0$ correct to 2 decimal places.

(6 marks)

This polynomial has a continuous graph: it has no jumps or breaks. If its value changes from positive to negative, the graph must pass through zero somewhere in between

Keep the full calculator value for each next iteration (use the Ans key). Round only the values you write down, otherwise rounding errors can build up

a) $f(1) = 1 + 2 - 5 = -2$, which is negative, and $f(2) = 4 + 4 - 5 = 3$, which is positive

a) The sign change means there is a root between $x = 1$ and $x = 2$

b) Rearrange to $x^2 + 2x = 5$, factorise the left side as $x(x+2) = 5$, then divide by $(x+2)$: $x = \frac{5}{x+2}$, as required

$$c) x_1 = \frac{5}{1.5 + 2} = 1.428571... = 1.4286 \text{ (4 dp)}$$

$$c) x_2 = \frac{5}{1.428571 + 2} = 1.458333... = 1.4583 \text{ (4 dp)}$$

$$c) x_3 = \frac{5}{1.458333 + 2} = 1.445783... = 1.4458 \text{ (4 dp)}$$

d) The iterates settle around 1.45; the exact root is $-1 + \sqrt{6} = 1.449489...$, so 1.45 to 2 dp

To check the 2-decimal-place rounding, $f(1.445) = -0.021975$ and $f(1.455) = 0.027025$. The positive root lies between these rounding boundaries, so it rounds to 1.45

12 The equation $x^3 - 4x - 3 = 0$ has a root between 2 and 3.

Use the iteration formula $x_{n+1} = \sqrt[3]{4x_n + 3}$ with $x_0 = 2.5$.

a. Work out x_1 , x_2 and x_3 . Give each answer to 4 decimal places.

b. By evaluating $f(x) = x^3 - 4x - 3$ at 2.295 and 2.305, show that the root is 2.30 correct to 2 decimal places.

(5 marks)

a)

$$x_1 = \sqrt[3]{(4 \times 2.5 + 3)} = \sqrt[3]{13}$$

$$x_1 = 2.3513$$

$$x_2 = \sqrt[3]{(4 \times 2.3513 + 3)}$$

$$x_2 = 2.3149$$

$$x_3 = \sqrt[3]{(4 \times 2.3149 + 3)}$$

$$x_3 = 2.3058$$

b)

$$f(2.295) = -0.092$$

$$f(2.305) = 0.027$$

Sign change, so the root is in $2.295 < x < 2.305$

$$\text{Root} = 2.30 \text{ (2 d.p.)}$$

13 $x^3 - 2x^2 - 1 = 0$ has a root between 2 and 3.

a. Show that the equation can be rearranged to give $x = 2 + \frac{1}{x^2}$.

b. Use the iteration $x_{n+1} = 2 + \frac{1}{x_n^2}$ with $x_0 = 2$ to find x_1 , x_2 and x_3 , to 4 decimal places.

c. Given that $f(2.205) = -0.0033$ and $f(2.215) = 0.0548$ where $f(x) = x^3 - 2x^2 - 1$, give the root to 2 decimal places.

(5 marks)

a)

$$\begin{array}{rcl} x^3 - 2x^2 - 1 & = & 0 \\ +1 + 2x^2 & & \\ \hline x^3 & = & 2x^2 + 1 \\ \div x^2 & & \\ x & = & \frac{2x^2 + 1}{x^2} \\ & & \\ x & = & 2 + \frac{1}{x^2} \end{array}$$

b)

$$x_1 = 2 + \frac{1}{2^2} = 2.25$$

$$x_2 = 2 + \frac{1}{2.25^2} = 2.1975$$

$$x_3 = 2 + \frac{1}{2.1975^2} = 2.2071$$

c)

$$f(2.205) < 0 \text{ and } f(2.215) > 0$$

Root lies between 2.205 and 2.215

$$\boxed{\text{Root} = 2.21 \text{ (2 d.p.)}}$$