



- 1 Prove algebraically that the difference between the squares of two consecutive integers is equal to the sum of the two integers.

(3 marks)

Let the integers be n and $n + 1$

$$= (n + 1)^2 - n^2$$

$$= n^2 + 2n + 1 - n^2$$

$$= 2n + 1$$

$$= n + (n + 1)$$

= the sum of the two integers

- 2 A regular hexagon is drawn inside a circle of radius r so that all six vertices lie on the circle. Each side of the hexagon is equal to r .

The perimeter of the hexagon is less than the circumference of the circle.

Use this to show that $\pi > 3$.

(3 marks)

$$\text{perimeter of hexagon} = 6 \times r = 6r$$

$$\text{circumference of circle} = 2\pi r$$

hexagon perimeter is less

$$= 6r < 2\pi r$$

divide both sides by $2r$ (positive)

$$= 3 < \pi$$

$\pi > 3$, as required

- 3 Prove that $x^2 - 6x + 11$ is greater than 0 for all values of x .

(3 marks)

complete the square

$$x^2 - 6x + 11 = (x - 3)^2 - 9 + 11$$

$$= (x - 3)^2 + 2$$

a squared number is never negative

$$= (x - 3)^2 \geq 0$$

$$= (x - 3)^2 + 2 \geq 2$$

so it is always greater than 0

- 4 Prove algebraically that the sum of the squares of any three consecutive integers is always 2 more than a multiple of 3.

(3 marks)

Let the integers be $n-1, n, n+1$

$$= (n-1)^2 + n^2 + (n+1)^2$$

$$= n^2 - 2n + 1 + n^2 + n^2 + 2n + 1$$

$$= 3n^2 + 2$$

$$= 3n^2 + 2, 2 \text{ more than a multiple of } 3$$

- 5 Prove algebraically that $(3n+1)^2 - (3n-1)^2$ is a multiple of 12 for every integer n .

(3 marks)

Expand both squares

$$(3n+1)^2 = 9n^2 + 6n + 1 \text{ and } (3n-1)^2 = 9n^2 - 6n + 1$$

Subtract

$$(9n^2 + 6n + 1) - (9n^2 - 6n + 1) = 12n$$

$12n$ is 12 multiplied by an integer, so it is a multiple of 12

Therefore $(3n+1)^2 - (3n-1)^2$ is a multiple of 12 for every integer n

- 6 Prove algebraically that the difference between the squares of any two consecutive odd numbers is a multiple of 8.

(4 marks)

Let the two consecutive odd numbers be $2n-1$ and $2n+1$, where n is an integer

$$(2n+1)^2 = 4n^2 + 4n + 1 \text{ and } (2n-1)^2 = 4n^2 - 4n + 1$$

$$\text{Difference} = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$$

$8n$ is 8 multiplied by an integer, so the difference is a multiple of 8

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a. Show that $(n + 3)^2 - (n - 2)^2 \equiv 5(2n + 1)$ b. Hence explain why the difference between $(n + 3)^2$ and $(n - 2)^2$ is always an odd multiple of 5, for every integer n .

(4 marks)

a) Expand

$$(n + 3)^2 = n^2 + 6n + 9 \text{ and } (n - 2)^2 = n^2 - 4n + 4$$

Subtract

$$(n^2 + 6n + 9) - (n^2 - 4n + 4) = 10n + 5$$

Factorise

$$10n + 5 = 5(2n + 1), \text{ as required}$$

b) For every integer n , $2n + 1$ is an odd number

So the difference is 5 multiplied by an odd number, which is an odd multiple of 5

8

Prove algebraically that $n^3 - n$ is a multiple of 6 for every positive integer n .

(4 marks)

$$n^3 - n = n(n^2 - 1)$$

$$= n(n - 1)(n + 1)$$

three consecutive integers: $(n - 1)$, n , $(n + 1)$

at least one is even, so the product is a multiple of 2

exactly one is a multiple of 3, so the product is a multiple of 3

multiple of 2 and 3, so a multiple of 6

9

Prove algebraically that the square of any odd number is 1 more than a multiple of 8.

(4 marks)

Let the odd number be $2n + 1$, where n is an integer

$$(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$$

 n and $n + 1$ are consecutive integers, so $n(n + 1)$ is even

$$\text{write } n(n + 1) = 2k \text{ for an integer } k$$

$$\text{Then } (2n + 1)^2 = 4 \times 2k + 1 = 8k + 1$$

 $8k$ is a multiple of 8, so the square of any odd number is 1 more than a multiple of 8

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Prove algebraically that the difference between the squares of any two odd numbers is a multiple of 8.

(6 marks)

Let the two odd numbers be $2a + 1$ and $2b + 1$, where a and b are integers (they need not be consecutive)

$$(2a + 1)^2 - (2b + 1)^2 = (4a^2 + 4a + 1) - (4b^2 + 4b + 1) = 4a^2 + 4a - 4b^2 - 4b$$

Factorise

$$= 4a(a + 1) - 4b(b + 1)$$

$a(a + 1)$ is a product of consecutive integers, so it is even

$$a(a + 1) = 2m \text{ for an integer } m. \text{ Similarly } b(b + 1) = 2k$$

$$\text{So the difference} = 4 \times 2m - 4 \times 2k = 8m - 8k = 8(m - k)$$

$m - k$ is an integer, so the difference between the squares of any two odd numbers is a multiple of 8

11

Prove that the product of four consecutive integers, plus 1, is always a perfect square.

(4 marks)

integers $n, n+1, n+2, n+3$: pair outer and inner

$$n(n + 3) = n^2 + 3n$$

$$(n + 1)(n + 2) = n^2 + 3n + 2$$

$$\text{let } m = n^2 + 3n$$

$$m(m + 2) + 1 = m^2 + 2m + 1$$

$$= (m + 1)^2$$

$$= (n^2 + 3n + 1)^2, \text{ a perfect square}$$