



- 1** AB is a diameter of a circle with centre O.
C is a point on the circle, not at A or B.
Prove that angle $ACB = 90^\circ$.

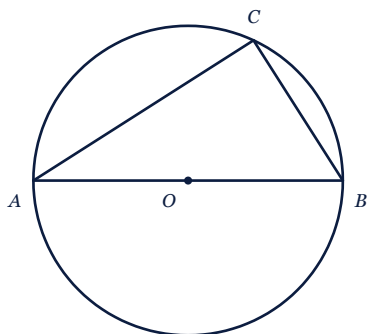


Diagram NOT accurately drawn

(3 marks)

Join O to C. $OA = OC$ and $OB = OC$ because all three are radii of the circle

Triangle OAC is isosceles, so angle $OAC = \text{angle } OCA$. Call this angle x

Triangle OBC is isosceles, so angle $OBC = \text{angle } OCB$. Call this angle y

In triangle ABC the three angles are angle $CAB = x$, angle $CBA = y$ and angle $ACB = x + y$

Angles in a triangle sum to 180° , so $x + y + (x + y) = 180^\circ$, which gives $2(x + y) = 180^\circ$

So $x + y = 90^\circ$, and therefore angle $ACB = 90^\circ$

This proves that the angle in a semicircle is a right angle

- 2 AB is a chord of a circle with centre O.
M is the point on AB such that OM is perpendicular to AB.
Prove that M is the midpoint of AB.

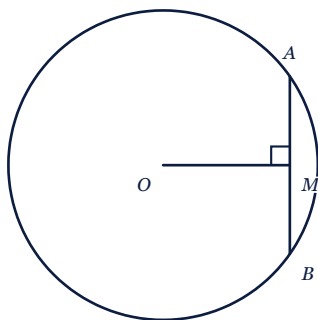


Diagram NOT accurately drawn

(4 marks)

If AB is a diameter, the perpendicular meets it at O, so $M = O$ and $AM = BM$ because both are radii. Otherwise, use the two right-angled triangles below

Join O to A and O to B

Angle $OMA = \text{angle } OMB = 90^\circ$ because OM is perpendicular to AB (given)

$OA = OB$ because both are radii of the circle. These are the hypotenuses of the two right-angled triangles

OM is common to both triangles, so $OM = OM$

So triangle OMA is congruent to triangle OMB: the triangles have the same shape and size. RHS means right angle, equal hypotenuse and one equal side

Corresponding sides of congruent triangles are equal, so $AM = BM$

Therefore M is the midpoint of AB, which proves that the perpendicular from the centre of a circle to a chord bisects the chord

- 3 P is a point outside a circle with centre O.
The two tangents from P touch the circle at A and at B.
Prove that $PA = PB$.

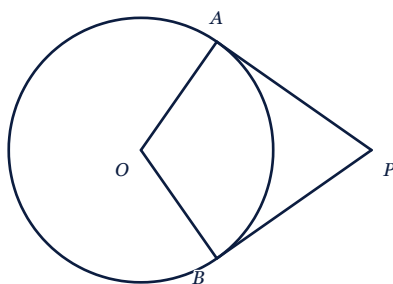


Diagram NOT accurately drawn

(4 marks)

Join O to A, O to B and O to P

Angle $OAP = \text{angle } OBP = 90^\circ$ because a tangent to a circle is perpendicular to the radius at the point of contact

$OA = OB$ because both are radii of the circle

OP is the hypotenuse of both right-angled triangles and is common, so $OP = OP$

So triangle OAP is congruent to triangle OBP (same shape and size), by RHS: a right angle, equal hypotenuse and one equal side

Corresponding sides of congruent triangles are equal, so $PA = PB$

This proves that the two tangents drawn to a circle from an external point are equal in length

- 4 A, B and P are points on a circle with centre O.
O lies inside triangle ABP.
Prove that $\text{angle AOB} = 2 \times \text{angle APB}$.

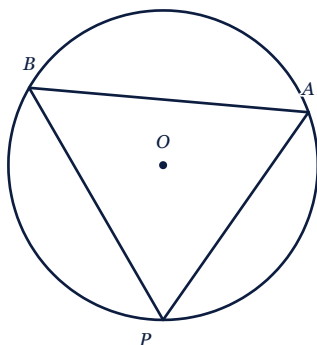


Diagram NOT accurately drawn

(4 marks)

Join P to O and extend the line to a point D on the other side of O

$OA = OP$ because both are radii, so triangle OAP is isosceles and $\text{angle OPA} = \text{angle OAP}$. Call this angle x

The exterior angle of a triangle equals the sum of the two interior opposite angles, so $\text{angle AOD} = \text{angle OPA} + \text{angle OAP} = 2x$

$OB = OP$ because both are radii, so triangle OBP is isosceles and $\text{angle OPB} = \text{angle OBP}$. Call this angle y

By the same exterior angle argument, $\text{angle BOD} = 2y$

$\text{Angle AOB} = \text{angle AOD} + \text{angle DOB} = 2x + 2y = 2(x + y)$

$\text{Angle APB} = \text{angle APD} + \text{angle DPB} = x + y$

Therefore $\text{angle AOB} = 2 \times \text{angle APB}$, which proves that the angle at the centre is twice the angle at the circumference standing on the same arc

- 5 A, B, P and Q are points on a circle with centre O.

P and Q are on the same side of the chord AB, and both are on the major arc AB.

You may use the theorem that the angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

Prove that $\text{angle APB} = \text{angle AQB}$.

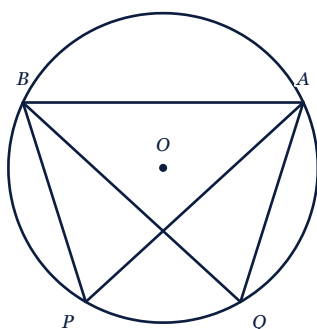


Diagram NOT accurately drawn

(4 marks)

Join O to A and O to B

P is on the major arc AB, so by the angle at the centre theorem, $\text{angle AOB} = 2 \times \text{angle APB}$

Q is on the same arc, so by the same theorem, $\text{angle AOB} = 2 \times \text{angle AQB}$

Therefore $2 \times \text{angle APB} = 2 \times \text{angle AQB}$

Dividing both sides by 2 gives $\text{angle APB} = \text{angle AQB}$

This proves that angles in the same segment of a circle are equal

6 ABCD is a cyclic quadrilateral in a circle with centre O.

You may use the theorem that the angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

Prove that $\text{angle } ABC + \text{angle } ADC = 180^\circ$.

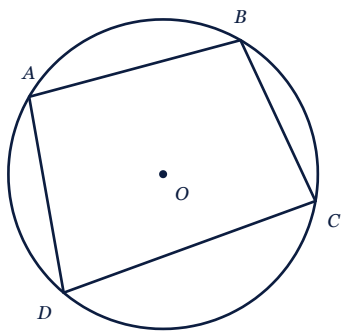


Diagram NOT accurately drawn

(4 marks)

A cyclic quadrilateral has all four vertices on a circle. Join O to A and O to C

There are two arcs from A to C

one through B and one through D. Let α be the central angle following arc AOC, and β the central angle following arc ABC. Angle ABC stands on arc AOC (the arc not containing B), so the angle-at-the-centre theorem gives $\alpha = 2 \times \text{angle } ABC$

Angle ADC stands on the other arc, ABC, so $\beta = 2 \times \text{angle } ADC$

Substitute into $\alpha + \beta = 360^\circ$: $2 \times \text{angle } ABC + 2 \times \text{angle } ADC = 360^\circ$

Divide by 2

$\text{angle } ABC + \text{angle } ADC = 180^\circ$. This also covers AC being a diameter: both central angles are then 180°

7 The line ST is a tangent to a circle with centre O, touching the circle at A.

AB is a chord of the circle.

C is a point on the major arc AB, so that C is in the alternate segment to angle TAB.

Prove that $\text{angle } TAB = \text{angle } ACB$.

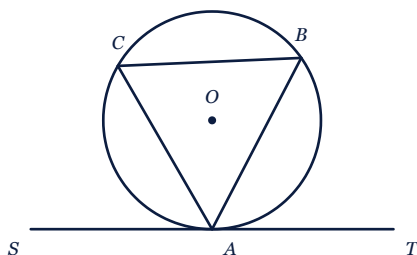


Diagram NOT accurately drawn

(4 marks)

If AB is a diameter, angle TAB is 90° (tangent perpendicular to radius) and angle ACB is 90° (angle in a semicircle), so they are equal. Otherwise use the construction below

Draw the diameter AD from A through the centre O, and join O to B

Angle DAT = 90° because a tangent is perpendicular to the radius (and so to the diameter) at the point of contact

Therefore $\text{angle } TAB = 90^\circ - \text{angle } OAB$

Angle ABO = 90° because it is the angle in a semicircle standing on the diameter AD

In triangle ABO the angles sum to 180° , so $\text{angle } AOB = 180^\circ - 90^\circ - \text{angle } OAB = 90^\circ - \text{angle } OAB$

So $\text{angle } TAB = \text{angle } AOB$

Angle AOB = angle ACB because both are angles in the same segment standing on the chord AB

Therefore $\text{angle } TAB = \text{angle } ACB$, which proves the alternate segment theorem

8

A, B and C are points on a circle.

DAE is a tangent to the circle at A.

B is on the opposite side of the chord AC from E.

Angle CAE = 63° and angle BAC = 44° .

Work out the size of

a. angle ABC,

b. angle ACB.

Give a reason for each stage of your working.

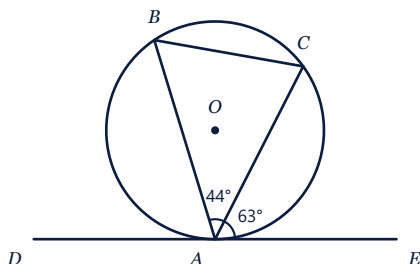


Diagram NOT accurately drawn

(4 marks)

a) Angle ABC = angle CAE = 63° because the angle between a tangent and a chord is equal to the angle in the alternate segment

b) Angles in a triangle add up to 180°
 Angle ACB = $180^\circ - 63^\circ - 44^\circ = 73^\circ$

9

AB and CD are chords of a circle with centre O.

AB = CD

M is the point on AB such that OM is perpendicular to AB.

N is the point on CD such that ON is perpendicular to CD.

Prove that OM = ON.

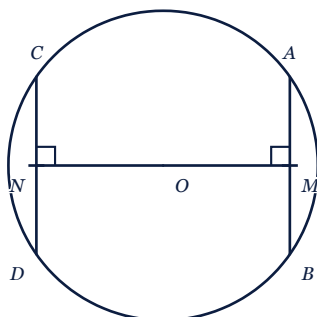


Diagram NOT accurately drawn

(5 marks)

If the equal chords are diameters, both perpendicular distances are zero and the result follows. Otherwise use the two right-angled triangles below

Join O to A and O to C

The perpendicular from the centre of a circle to a chord bisects the chord, so $AM = AB \div 2$ and $CN = CD \div 2$

AB = CD (given), so $AM = CN$

OA = OC because both are radii of the circle

Triangle OMA has a right angle at M, so by Pythagoras, $OM^2 = OA^2 - AM^2$

Triangle ONC has a right angle at N, so by Pythagoras, $ON^2 = OC^2 - CN^2$

Since OA = OC and $AM = CN$, it follows that $OM^2 = ON^2$

Distances are nonnegative, so $OM = ON$. This proves that equal chords of a circle are equidistant from the centre

Alternatively, triangles OMA and ONC are congruent (RHS

right angles at M and N, hypotenuses OA = OC, and $AM = CN$), so $OM = ON$

- 10 A, B and C are points on a circle with centre O. Angle ACB is acute. TA is a tangent to the circle at A, with T and C on opposite sides of the line AB. Prove that angle TAB = angle ACB.

(You may use the fact that the angle at the centre is twice the angle at the circumference.)

(4 marks)

$$\begin{aligned}
 \text{Let angle } OAB &= y \\
 OAT &= 90^\circ \text{ (tangent } \perp \text{ radius)} \\
 TAB &= 90^\circ - y \\
 OA &= OB \text{ (radii) so } OBA = y \\
 AOB &= 180^\circ - 2y \text{ (angles in a triangle)} \\
 ACB &= \frac{1}{2} \times (180^\circ - 2y) \\
 ACB &= 90^\circ - y \text{ (angle at centre} = 2 \times \text{angle at circumference)} \\
 \boxed{TAB = ACB = 90^\circ - y}
 \end{aligned}$$

- 11 A, B and C are points on a circle with centre O. PA and PB are tangents to the circle from the point P. C lies on the major arc AB. Angle APB = 42° .

Work out the size of angle ACB. Give a reason for each stage of your working.

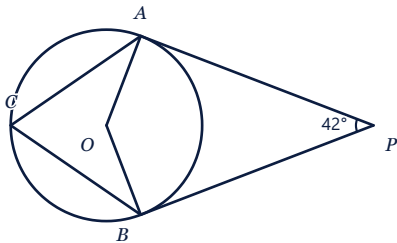


Diagram NOT accurately drawn

(4 marks)

$$\begin{aligned}
 \text{Angle } OAP &= \text{angle } OBP = 90^\circ \text{ because a tangent is perpendicular to the radius at the point of contact} \\
 \text{Angles in a quadrilateral add up to } 360^\circ, \text{ so angle } AOB &= 360^\circ - 90^\circ - 90^\circ - 42^\circ = 138^\circ \\
 \text{The angle at the centre is twice the angle at the circumference standing on the same arc} \\
 \text{Angle } ACB &= 138^\circ \div 2 = 69^\circ \\
 \boxed{69^\circ}
 \end{aligned}$$

- 12 A, B, C and D are points on a circle, in that order, so ABCD is a cyclic quadrilateral. TA is a tangent to the circle at A. T and D are on opposite sides of the line AB. Angle TAB = 48° . Angle BDC = 35° .

Work out the size of angle ABC. Give a reason for each stage of your working.

(4 marks)

$$\begin{aligned}
 \text{Alternate segment theorem} \\
 AOB &= TAB = 48^\circ \\
 AOC &= 48^\circ + 35^\circ = 83^\circ \\
 \text{Opposite angles in a cyclic quad add to } 180^\circ \\
 ABC &= 180^\circ - 83^\circ \\
 \boxed{97^\circ}
 \end{aligned}$$

- 13 A, B, C and D are points on a circle, in that order.
The chords AC and BD intersect at the point E inside the circle.
Prove that triangle ABE is similar to triangle DCE.

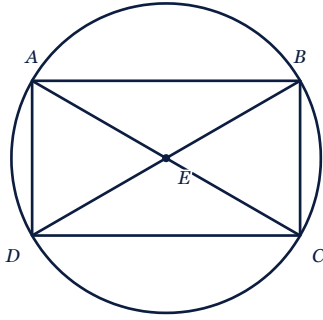


Diagram NOT accurately drawn

(4 marks)

Angle BAE = angle BAC and angle COE = angle COB. Angle BAC = angle BOC because angles in the same segment, standing on chord BC, are equal. So angle BAE = angle COE

Angle ABE = angle ABD and angle OCE = angle ACO. Angle ABD = angle ACO because angles in the same segment, standing on chord AD, are equal. So angle ABE = angle OCE

Angle AEB = angle DEC because vertically opposite angles are equal

The triangles have equal angles, so triangle ABE is similar to triangle DCE

Similar triangles have the same shape, though their sizes can differ. AA means that two matching angles are sufficient: the third also matches because each triangle's angles total 180°

- 14 A, B and C are points on a circle with centre O.
PA and PB are tangents to the circle from the point P.
C lies on the major arc AB.
Angle APB = x° .

Find the size of angle ACB in terms of x. Give your answer in its simplest form and give a reason for each stage of your working.

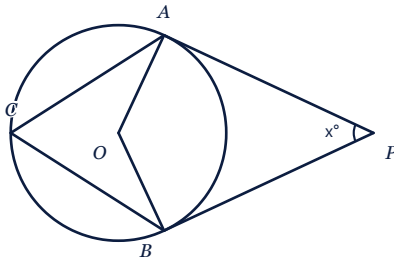


Diagram NOT accurately drawn

(5 marks)

Angle OAP = angle OBP = 90° because a tangent is perpendicular to the radius at the point of contact

Angles in a quadrilateral add up to 360° , so angle AOB = $360^\circ - 90^\circ - 90^\circ - x^\circ = (180 - x)^\circ$

The angle at the centre is twice the angle at the circumference standing on the same arc

Angle ACB = $(180 - x)^\circ \div 2 = (90 - \frac{x}{2})^\circ$

$$(90 - \frac{x}{2})^\circ$$

- 15 A, B, C and D are points on the circumference of a circle, in that order, so ABCD is a cyclic quadrilateral.

$$\text{Angle DAB} = (3x + 10)^\circ$$

$$\text{Angle BCD} = (5x - 30)^\circ$$

$$\text{Angle ABC} = (2x + 25)^\circ$$

Work out the size of angle ADC. Give a reason for each stage of your working.

(4 marks)

Opposite angles in a cyclic quad add to 180°

$$\begin{array}{rcl} (3x + 10) + (5x - 30) & = & 180 \\ 8x - 20 & = & 180 \\ +20 & & +20 \\ 8x & = & 200 \\ \div 8 & & \div 8 \\ x & = & 25 \\ \text{ABC} & = & 2 \times 25 + 25 = 75^\circ \end{array}$$

ADC is opposite ABC

$$\text{ADC} = 180^\circ - 75^\circ$$

$$\boxed{105^\circ}$$

- 16 ABCD is a cyclic quadrilateral.

$$\text{angle BAD} : \text{angle BCD} = 4 : 5$$

$$\text{angle ABC} = \text{angle BAD} + 30^\circ$$

Work out the size of angle ADC. Give a reason for each stage of your working.

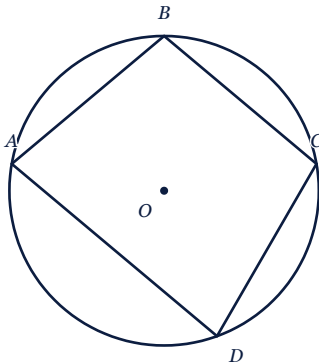


Diagram NOT accurately drawn

(5 marks)

Opposite angles of a cyclic quadrilateral add up to 180° , so angle BAD + angle BCD = 180°

The ratio 4

$$5 \text{ has 9 parts, so one part} = 180^\circ \div 9 = 20^\circ$$

$$\text{Angle BAD} = 4 \times 20^\circ = 80^\circ \text{ and angle BCD} = 5 \times 20^\circ = 100^\circ$$

$$\text{Angle ABC} = 80^\circ + 30^\circ = 110^\circ$$

Opposite angles of a cyclic quadrilateral add up to 180° , so angle ADC = $180^\circ - 110^\circ = 70^\circ$

$$\boxed{70^\circ}$$

- 17 A, B and C are points on a circle. AB = AC.

The line TAS is a tangent to the circle at A. T and C are on opposite sides of the line AB.

Prove that TS is parallel to BC.

(4 marks)

$$\text{TAB} = \text{ACB} \text{ (alternate segment theorem)}$$

$$\text{ABC} = \text{ACB} \text{ (AB = AC, isosceles triangle)}$$

$$\text{So TAB} = \text{ABC}$$

Equal alternate angles

$$\boxed{\text{TS is parallel to BC}}$$