Stationary points and turning points

Worked solutions

GCSE Maths topic practice

This topic online

1 The curve $y = x^2 - 6x + 1$ has one turning point.

a. Use differentiation to find the coordinates of the turning point.

b. State whether the turning point is a maximum or a minimum.

(3 marks)

a)

$\frac{dy}{dx} = 2x - 6$

$2x - 6 = 0$ (+6, ÷2)

$x = 3$ (+6, ÷2)

$y = 3^2 - 6(3) + 1 = -8$

$(3, -8)$

b)

$+x^2$ so a U shape

minimum

2 A curve has equation $y = 5 + 8x - 2x^2$

Find the coordinates of the maximum point of the curve.

(3 marks)

$\frac{dy}{dx} = 8 - 4x$

$8 - 4x = 0$ (+4x, ÷4)

$x = 2$ (+4x, ÷4)

$y = 5 + 8(2) - 2(2)^2 = 5 + 16 - 8$

$-x^2$ so an n shape: maximum

$(2, 13)$

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3 The curve C has equation $y = x^3 - 3x + 6$

The point P has coordinates (1, 4).

a. Show that P is a stationary point of C.

b. State, with a reason, whether P is a maximum point or a minimum point.

(3 marks)

a)

$$\frac{dy}{dx} = 3x^2 - 3$$

$x = 1$: $\frac{dy}{dx} = 3(1) - 3 = 0$

$x = 1$: $y = 1 - 3 + 6 = 4$

P is on C and the gradient there is 0, so P is a stationary point

b)

$x = 0$: $\frac{dy}{dx} = -3$ (going down)

$x = 2$: $\frac{dy}{dx} = 9$ (going up)

Down then up, so P is a minimum

minimum

4 The curve $y = x^2 + kx + 7$, where k is a constant, has its minimum point at $x = 4$

a. Find the value of k.

b. Find the y-coordinate of the minimum point.

(4 marks)

a)

$$\frac{dy}{dx} = 2x + k$$

Gradient is 0 at $x = 4$

$2(4) + k = 0$

$8 + k = 0$

$k = -8$

b)

$y = x^2 - 8x + 7$

$x = 4$: $y = 16 - 32 + 7$

$y = -9$

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5 A delivery company works out that the cost, £C, of fuel for a journey when a van travels at an average speed of x km/h is given by

$C = 0.02x^2 - 2x + 60$ for $20 \leq x \leq 80$

a. Find the speed, in km/h, that makes the fuel cost as small as possible.

b. Find this smallest fuel cost, in pounds.

(4 marks)

a)

$$\frac{dC}{dx} = 0.04x - 2$$

$$0.04x - 2 = 0$$

$$0.04x = 2 \quad (+2)$$

$$x = 50 \quad (\div 0.04)$$

$+x^2$ so this is the minimum

50 km/h

b)

$$C = 0.02 \times 50^2 - 2 \times 50 + 60$$

$$= 50 - 100 + 60$$

£10

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6 Priya runs a market stall selling strawberries. When she sells x kg of strawberries in a day, her profit, £P, is given by

$P = -0.5x^2 + 30x - 100$

a. Find the number of kilograms Priya should sell in a day to make the greatest profit.

b. Find the greatest possible daily profit, in pounds.

(4 marks)

a)

$\frac{dP}{dx} = -x + 30$

$-x + 30 = 0$

$+x \qquad x = 30 \qquad +x$

$-x^2$ so an n shape: this is the maximum

30 kg

b)

$P = -0.5 \times 30^2 + 30 \times 30 - 100$

$= -450 + 900 - 100$

£350

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7 The curve C has equation $y = x + \frac{9}{x}$ for $x > 0$

a. Find the coordinates of the stationary point of C.

b. State whether this stationary point is a maximum or a minimum.

(4 marks)

a)

$$y = x + 9x^{-1}$$

$$\frac{dy}{dx} = 1 - 9x^{-2} = 1 - \frac{9}{x^2}$$

$$1 - \frac{9}{x^2} = 0$$

$$\frac{9}{x^2} = 1$$

$$x^2 = 9$$

$x = 3$ ($x > 0$, so not -3)

$$y = 3 + \frac{9}{3} = 6$$

$\boxed{(3, 6)}$

b)

$x = 1$: $\frac{dy}{dx} = 1 - 9 = -8$ (going down)

$x = 9$: $\frac{dy}{dx} = 1 - \frac{9}{81} = \frac{8}{9}$ (going up)

Down then up, so a minimum

$\boxed{\text{minimum}}$

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8 The curve C has equation $y = x^3 - 6x^2 + 9x + 1$

a. Find the x-coordinates of the two stationary points of C.

b. Write down the coordinates of the maximum point of C.

c. Write down the coordinates of the minimum point of C.

(5 marks)

a)

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$3x^2 - 12x + 9 = 0$$

$\div 3$ $\qquad$ $\div 3$

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$x = 1$ or $x = 3$

b)

$x = 1$: $y = 1 - 6 + 9 + 1 = 5$

$x = 3$: $y = 27 - 54 + 27 + 1 = 1$

$+x^3$ cubic: max first, then min

maximum (1, 5)

c)

minimum (3, 1)

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9 A farmer has 120 m of fencing. She uses all of it to make a rectangular pen against a long straight stone wall, so the fencing is needed on three sides only.

The two sides at right angles to the wall are each x metres long.

a. Write down an expression, in terms of x, for the area A m^2 of the pen. Give your answer in the form $A = px - qx^2$.

b. Use differentiation to find the value of x that gives the greatest area.

c. Find the greatest possible area of the pen, in m^2.

(5 marks)

a)

Side along the wall = $120 - 2x$

$A = x(120 - 2x)$

$\boxed{A = 120x - 2x^2}$

b)

$$\frac{dA}{dx} = 120 - 4x$$

$$120 - 4x = 0$$

$+4x$ $\quad 4x = 120 \quad +4x$

$\div 4$ $\quad x = 30 \quad \div 4$

$-x^2$ so this is the maximum

$\boxed{x = 30}$

c)

$A = 120(30) - 2(30)^2 = 3600 - 1800$

$\boxed{1800 \text{ m}^2}$

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10 An open box is made from a square sheet of card of side 24 cm. A square of side x cm is cut from each corner and the sides are folded up.

The volume, V cm^3, of the box is given by

$V = 4x^3 - 96x^2 + 576x \quad \text{for } 0 < x < 12$

a. Find an expression for $\frac{dV}{dx}$

b. Find the value of x that gives the greatest possible volume.

c. Find the greatest possible volume of the box, in cm^3. (5 marks)

a)

$$\frac{dV}{dx} = 12x^2 - 192x + 576$$

$\boxed{12x^2 - 192x + 576}$

b)

$$\begin{aligned} 12x^2 - 192x + 576 &= 0 \\ \div 12 \qquad x^2 - 16x + 48 &= 0 \qquad \div 12 \\ (x - 4)(x - 12) &= 0 \end{aligned}$$

$x = 4$ or $x = 12$

x must be less than 12, and $+x^3$ cubic: first stationary point is the max

$\boxed{x = 4}$

c)

$$V = 4(4)^3 - 96(4)^2 + 576(4)$$

$$= 256 - 1536 + 2304$$

$\boxed{1024 \text{ cm}^3}$

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